Solutions

MAT 296, Fall 2015, Exam #1

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(Please print)

- **You must show the details of your work to receive credit.**
- Simplify your answers whenever possible. Keep your answers exact: fractions, not decimals.
- Calculators, phones and other electronic devices **may not be used or be available for use during the exam.**

#	Points	Score
1	30	
2	30	
3	20	
4	10	
5	10	
Total	100	

1. Evaluate the following integrals.

(a) $I = \int \frac{x^2+1}{x^2(x-1)}\,dx$

$$\frac{x^2+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$= \frac{Ax(x-1) + B(x-1) + Cx^2}{x^2(x-1)}$$

$Ax(x-1) + B(x-1) + Cx^2$

$Ax^2 - Ax + Bx - B + Cx^2$

$x^2: \ A + C = 1$

$x: \ -A + B = 0$

$1: \ -B = 1$

$B = -1$

$A = B = -1$

$C = 2$

OR

$B = \frac{0^2+1}{0-1} = -1$

$C = \frac{1^2+1}{1^2} = 2$

$$\int \frac{x^2+1}{x^2(x-1)}\,dx = \int \frac{-1}{x} + \frac{-1}{x^2} + \frac{2}{x-1}\,dx$$

$-\ln|x| + \frac{1}{x} + 2\ln|x-1| + K$

$\frac{1}{x} - \ln|x| + 2\ln|x-1| + K$

$$\boxed{\frac{1}{x} + \ln\left|\frac{(x-1)^2}{x}\right| + K}$$

(b) $I = \int \frac{x+4}{x^3+4x}\,dx$

$$\frac{x+4}{x^3+4x} = \frac{x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2+4) + x(Bx+C)}{x(x^2+4)}$$

$$A(x^2+4) + x(Bx+C)$$
$$Ax^2 + 4A + Bx^2 + Cx$$

x^2: $A + B = 0$

x: $C = 1$ $\qquad C = 1$

1: $4A = 4 \longrightarrow A = 1$

so $B = -1$

OR

$$A = \frac{0+4}{0^2+4} = 1$$

$$\int \frac{1}{x}\,dx = \ln|x|$$

$$\int \frac{-x}{x^2+4}\,dx = \frac{-1}{2}\int \frac{1}{u}\,du$$

$u = x^2+4$ $\qquad = -\frac{1}{2}\ln|u|$

$du = 2x\,dx$ $\qquad = \frac{-1}{2}\ln|x^2+4|$

$dx = \frac{du}{2x}$ $\qquad = \ln\left|\frac{1}{\sqrt{x^2+4}}\right|$

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1/4}{\left(\frac{x}{2}\right)^2+1}\,dx$$

$$\frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1} = \frac{1}{4}\cdot 2\arctan(x/2)$$

$$= \frac{1}{2}\arctan 2(x/2)$$

$$\int \frac{x+4}{x^3+4x}\,dx = \int \frac{1}{x} + \frac{-x+1}{x^2+4}\,dx$$

$$= \int \frac{1}{x} + \frac{-x}{x^2+4} + \frac{1}{x^2+4}\,dx$$

$$= \ln|x| + \ln\left|\frac{1}{\sqrt{x^2+4}}\right| + 1/2 \arctan(x/2) + K$$

$$= \boxed{\frac{\arctan(x/2)}{2} + \ln\left|\frac{x}{\sqrt{x^2+4}}\right| + K}$$

2. Determine which of the following improper integrals are convergent, which are divergent, and evaluate the convergent ones.

(a) $\int_0^\infty xe^{-x^2}\,dx$

$$\int xe^{-x^2}dx = \frac{-1}{2}\int e^u du$$

$u = -x^2$

$du = -2x\,dx$

$dx = \frac{du}{-2x}$

$$= \frac{-1}{2}e^u$$

$$= -\frac{e^{-x^2}}{2}$$

$$\lim_{b\to\infty}\int_0^b xe^{-x^2}dx = \lim_{b\to\infty} -\frac{e^{-x^2}}{2}\Big|_0^b$$

$$= \lim_{b\to\infty} -\frac{e^{-b^2}}{2} - -\frac{e^0}{2}$$

$$= 0 + 1/2$$

$$= 1/2$$

So the integral converges to 1/2.

(b) $\int_1^4 \frac{1}{(x-2)^2}\,dx$

$$\int \frac{dx}{(x-2)^2} = \frac{-1}{x-2}$$

$$\int_1^4 \frac{dx}{(x-2)^2} = \lim_{b\to 2}\int_1^b \frac{dx}{(x-2)^2} + \lim_{b\to 2}\int_b^4 \frac{dx}{(x-2)^2}$$

$$\lim_{b\to 2} \frac{-1}{x-2}\Big|_1^b \qquad \lim_{b\to 2}\frac{-1}{x-2}\Big|_b^4$$

$$\lim_{b\to 2}\frac{-1}{b-2} - \frac{-1}{1-2} \qquad \frac{-1}{4-2} - \lim_{b\to 2}\frac{-1}{b-2}$$

DNE DNE

So the integral does not converge.

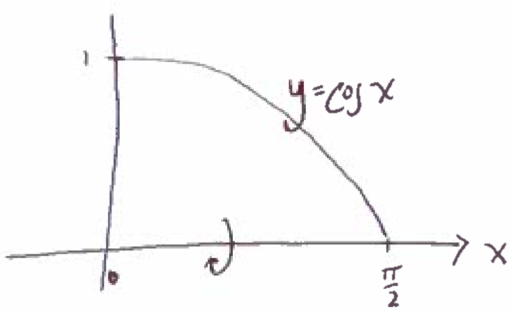
3. (a) Find the length of the arc along the curve $y = 2x^{\frac{3}{2}}$ from $(0, 0)$ to $(4, 16)$.

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$y = 2x^{3/2}$$
$$y' = 3x^{1/2}$$

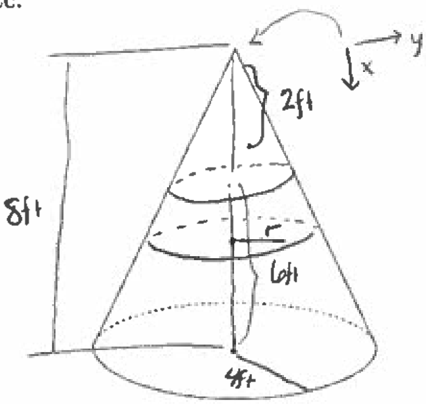
$$L = \int_0^4 \sqrt{1 + (3x^{1/2})^2}\, dx$$
$$= \int_0^4 \sqrt{1 + 9x}\, dx$$
$$\frac{1}{9} \cdot \frac{2}{3} (1 + 9x)^{3/2} \Big|_0^4$$
$$\frac{2}{27} \cdot (1 + 9x)^{3/2} \Big|_0^4$$
$$\frac{2}{27} (37)^{3/2} - \frac{2}{27} = \boxed{\frac{2}{27} (37\sqrt{37} - 1)}$$

(b) Set up, but do not evaluate, an explicit integral for the surface area of the figure generated by revolving about the x-axis the curve $y = \cos x$, $0 \leq x \leq \frac{\pi}{2}$.



$$SA = 2\pi \int_0^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin^2 x}\, dx$$

4. The conical tank shown below has height 8ft and radius 4ft and is filled to a level 2ft below the top with a liquid with density 2lb/ft^3. How much work does it take to pump all the liquid out through the top of the tank? For your final answer, you need not multiply out numbers like 8^3, etc.

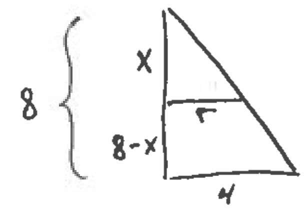


$$\text{Work} = \sum \text{Force} \cdot \text{distance}$$

Here force is a weight

$$\text{Weight} = \text{Volume} \cdot \text{weight density} = V \cdot \rho$$

$$V = \pi r^2 dx = \pi \left(\frac{x}{2}\right)^2 dx = \frac{\pi}{4} x^2 dx$$



$$\frac{x}{r} = \frac{8}{4} = 2$$

$$r = x/2$$

Distance slice moves is x

$$\text{Work} = \int_2^8 \left(\frac{\pi}{4} x^2 \rho\right) \cdot x \, dx$$

$$= \frac{\pi\rho}{4} \int_2^8 x^3 \, dx$$

$$= \frac{\pi\rho}{4} \left(\frac{x^4}{4} \Big|_2^8 \right)$$

$$= \frac{\pi\rho}{4} \left(\frac{8^4}{4} - \frac{2^4}{4} \right)$$

$$= \frac{\pi\rho}{16} \left(8^4 - 2^4 \right)$$

$$= \frac{\pi\rho}{16} \left((2^3)^4 - 2^4 \right)$$

$$= \frac{\pi\rho}{16} \left(2^{12} - 2^4 \right)$$

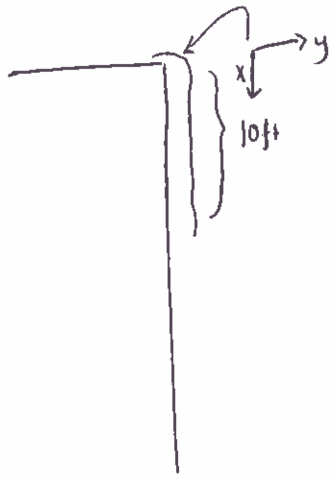
$$= \frac{\pi\rho\, 2^4}{16} \left(2^8 - 1 \right)$$

$$= \pi\rho \left(2^8 - 1 \right)$$

$$= \pi\rho \, (256-1)$$

$$\boxed{255\pi\rho \text{ ft}\cdot\text{lb}}$$

5. A chain 10ft long that weighs 5 pounds per foot is hanging from the top of a 45 foot building. How much work is done in pulling the chain all the way up to the top of the building?



Weight whole chain:

$10 ft\,(5\,lb/ft) = 50\,lb$

$$\text{Work} = \sum_i \text{Force} \cdot \text{distance}$$

Here force is a weight

$$F = \text{weight} = 50 - 5x$$

distance : dx

$$\text{Work} = \int_0^{10} 50 - 5x \, dx$$

$$= \left(50x - 5\frac{x^2}{2}\right)\Big|_0^{10}$$

$$\left(50(10) - 5\frac{(10)^2}{2}\right) - 0$$

$$500 - \frac{500}{2}$$

$$500 - 250$$

$$\boxed{250\ ft \cdot lb}$$