Solutions

MAT 296, Fall 2015, Exam #1

Name: Caleb McWhorter

(Please print)

- **You must show the details of your work to receive credit.**
- Simplify your answers whenever possible. Keep your answers exact: fractions, not decimals.
- Calculators, phones and other electronic devices **may not be used or be available for use during the exam.**

#	Points	Score
1	30	
2	30	
3	20	
4	10	
5	10	
Total	100	

1. Evaluate the following integrals.

(a) $I = \int \frac{x^2+1}{x^2(x-1)}\,dx$

$$\frac{x^2+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$= \frac{Ax(x-1) + B(x-1) + Cx^2}{x^2(x-1)}$$

x^2: $A + C = 1$

x: $B - A = 0$

1: $-B = 1$

So $B = -1$ then as $B = A$, $A = -1$ and as $A + C = 1$, $C = 2$

$$\int \frac{x^2+1}{x^2(x-1)}\,dx = \int \frac{-1}{x} + \frac{-1}{x^2} + \frac{2}{x-1}\,dx$$

$$-\ln|x| + \frac{1}{x} + 2\ln|x-1| + K$$

$$\frac{1}{x} + \ln|(x-1)^2| - \ln|x| + K$$

$$\frac{1}{x} + \ln\left|\frac{(x-1)^2}{x}\right| + K$$

(b) $I = \int \frac{x+4}{x^3+4x}\,dx$

$$\frac{x+4}{x^3+4x} = \frac{x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2+4) + x(Bx+C)}{x(x^2+4)}$$

x^2: $A+B=0$

x: $C=1$

1: $4A=4$

So $C=1$, $A=1$, and as $A+B=0$, $B=-1$.

$$\int \frac{x+4}{x^3+4x}\,dx = \int \frac{1}{x} + \frac{-x+1}{x^2+4}\,dx$$

$$= \int \frac{1}{x} + \frac{-x}{x^2+4} + \frac{1}{x^2+4}\,dx$$

$$\int \frac{1}{x}\,dx = \ln|x|$$

$$\int \frac{-x}{x^2+4}\,dx = \int \frac{-x}{u}\,\frac{du}{2x} = \frac{-1}{2}\int \frac{1}{u}\,du$$

$$= -\frac{1}{2}\ln|u|$$

$$= -\frac{1}{2}\ln|x^2+4|$$

$u = x^2+4$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1/4}{x^2+4 \cdot 1/4}\,dx = \int \frac{1/4}{\frac{x^2}{4}+1}\,dx$$

$$= \frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1} = \frac{1}{4} \cdot 2\arctan(x/2) = \frac{1}{2}\arctan(x/2)$$

So $\int \frac{x+4}{x^3+4x}\,dx = \ln|x| - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\arctan(x/2) + K$

$$= \ln\left|\frac{x}{\sqrt{x^2+4}}\right| + \frac{1}{2}\arctan(x/2) + K$$

2. Determine which of the following improper integrals are convergent, which are divergent, and evaluate the convergent ones.

(a) $\int_0^\infty xe^{-x^2}\,dx$

$$\int xe^{-x^2}dx = \int xe^u \frac{du}{-2x} = \frac{-1}{2}\int e^u du = \frac{-e^u}{2} = \frac{-e^{-x^2}}{2}$$

$u = -x^2$

$du = -2x\,dx$

$dx = \frac{du}{-2x}$

$$\lim_{b\to\infty}\int_0^b xe^{-x^2}dx = \lim_{b\to\infty} \frac{-e^{-x^2}}{2}\Big|_0^b = \lim_{b\to\infty} \frac{-e^{-b^2}}{2} - \frac{-e^0}{2}$$

$$= 0 - \frac{-1}{2}$$

$$= 1/2$$

(b) $\int_1^4 \frac{1}{(x-2)^2}\,dx$

$$\int \frac{1}{(x-2)^2}dx = \frac{-1}{x-2}$$

$$\lim_{b\to2^-}\int_1^b \frac{1}{(x-2)^2}dx = \lim_{b\to2^-} \frac{-1}{x-2}\Big|_1^b = \lim_{b\to2^-} \frac{-1}{b-2} - \frac{-1}{1-2} = \infty$$

So this integral diverges. We need not even consider

$$\lim_{b\to2^+}\int_b^2 \frac{1}{(x-2)^2}dx$$

3. (a) Find the length of the arc along the curve $y = 2x^{\frac{3}{2}}$ from $(0,0)$ to $(4,16)$.

$y = 2x^{3/2}$

$y' = 3x^{1/2}$

$(y')^2 = 9x$

$$L = \int_0^4 \sqrt{1+(y')^2}\,dx$$

$$= \int_0^4 \sqrt{1+9x}\,dx$$

$u = 1+9x$ if $x=0$, $u=1$

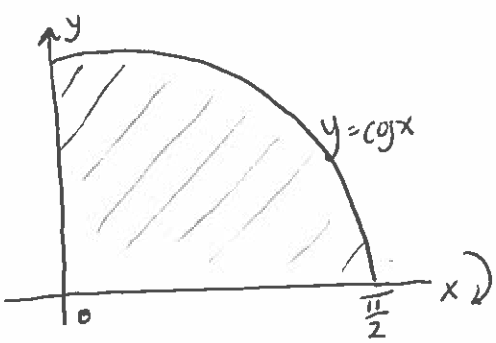
$du = 9\,dx$ if $x=4$, $u=37$

$$= \frac{1}{9}\int_1^{37} u^{1/2}\,du = \frac{2}{27} u^{3/2}\Big|_1^{37}$$

$$= \frac{2}{27}\sqrt{37^3} - \frac{2}{27}$$

$$= \frac{2(\sqrt{50653}-1)}{27} \approx 16.597$$

(b) Set up, but do not evaluate, an explicit integral for the surface area of the figure generated by revolving about the x-axis the curve $y = \cos x$, $0 \le x \le \frac{\pi}{2}$.



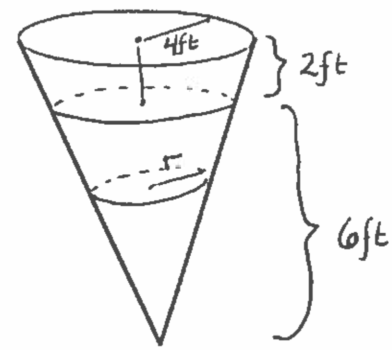
$$SA = 2\pi \int_0^{\frac{\pi}{2}} \cos x \sqrt{1+\sin^2 x}$$

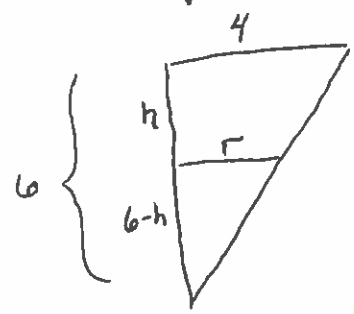
$y = \cos x$

$y' = -\sin x$

$(y')^2 = \sin^2 x$

4. The conical tank shown below has height 8ft and radius 4ft and is filled to a level 2ft below the top with a liquid with density 2lb/ft^3. How much work does it take to pump all the liquid out through the top of the tank? For your final answer, do not multiply out numbers like, for example, 8^3, etc.





$$\frac{r}{6-h} = \frac{4}{6}$$

$$\boxed{r = \frac{4}{6}(6-h)}$$

$$r = 4 - \frac{4}{6}h$$

$$\text{Weight} = \text{Volume} \cdot \text{weight density}$$

$$= \pi r^2 dh \cdot 2$$

$$= 2\pi r^2\, dh$$

$$= 2\pi \left(\frac{4}{6}(6-h)\right)^2 dh$$

$$\text{Work} = \text{Force} \cdot \text{distance} = \text{Weight} \cdot \text{distance} = 2\pi\left(\frac{4}{6}(6-h)\right)^2 dh \cdot h$$

$$\text{Total Work} = \int_2^8 2\pi \left(\frac{4}{6}(6-h)\right)^2 h\, dh$$

$$= 2\pi \int_2^8 \frac{4^6}{6^2}\, h(6-h)^2\, dh$$

$$= \frac{2\pi\, 2^{12}}{2^2 \cdot 3^2} \int_2^8 h(6-h)^2\, dh$$

$$= \frac{2^{11}\pi}{9} \int_2^8 h(6-h)^2\, dh$$

$$= \frac{2^{11}\pi}{9} \int_2^8 h(36 + h^2 - 12h)\, dh$$

$$= \frac{2^{11}\pi}{9} \int_2^8 h^3 - 12h^2 + 36h\, dh$$

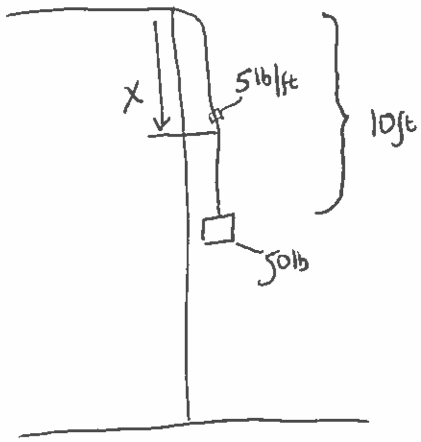
$$= \frac{2^{11}\pi}{9} \cdot \left(\frac{h^4}{4} - 4h^3 + 18h^2\right)\Bigg|_2^{2^3}$$

$$= \frac{2^{11}\pi}{9} \cdot \left[\left(\frac{2^{12}}{2^2} - 2^2 \cdot 2^9 + 18 \cdot 2^6\right) - \left(\frac{2^4}{2^2} - 2^2 \cdot 2^3 + 18 \cdot 2^2\right)\right]$$

$$= \frac{2^{11}\pi}{9}\left(2^{10} - 2^{11} + 18 \cdot 2^6 - 2^2 + 2^5 - 18 \cdot 2^2\right)$$

$$= \frac{57344\pi}{3} \text{ ft}\cdot\text{lb}$$

5. A chain 10ft long that weighs 5 pounds per foot is hanging from the top of a 45 foot building. At the end of the chain is a pail of cement weighing 50 pounds. How much work is done in pulling the chain (and pail) all the way up to the top of the building?



Weight of chain at height x: $5dx$

Work to lift this piece to top: $5dx \cdot x$

Work to lift chain: $\int_0^{10} 5x\,dx = 5\int_0^{10} x = 5 \cdot \frac{x^2}{2}\Big|_0^{10} = 5(50 - 0)$

$= 5(50) = 250$ ft·lb

Work to lift bucket: 50lb · 10ft = 500 ft·lb

Total Work = 250 ft·lb + 500 ft·lb = 750 ft·lb