Solutions

MAT 296, Fall 2015, Exam #3 Name: Caleb McWhorter

- **You must show the details of your work to receive credit.**
- **Simplify your answers** whenever possible. Keep your answers exact: fractions, not decimals.
- Calculators, phones and other electronic devices may not **be used or be available for use during the exam.**

#	Points	Score
1	15	
2	20	
3	30	
4	20	
5	15	
Total	100	

1. Determine whether the following sequences are convergent or divergent, and find the limit (in simplified form!) of the convergent sequences.

(a) $a_n = 2 + \left(-\frac{1}{3}\right)^n$.

$$a_n = 2 + \left(\frac{-1}{3}\right)^n = 2 + \frac{(-1)^n}{3^n}$$

$$\lim_{n\to\infty} \frac{(-1)^n}{3^n} = 0 \quad \text{So} \quad \boxed{\lim_{n\to\infty} a_n = 2}$$

(b) $a_n = \dfrac{n^2}{\sqrt{n^3+9n}}$

$$\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+9n}} \geq \lim_{n\to\infty} \frac{n^2}{\sqrt{n^3}} = \lim_{n\to\infty} n^{1/2} = \infty$$

$$\text{So} \quad \boxed{\lim_{n\to\infty} a_n = \infty \text{ (DNE)}}$$

(c) $a_n = \ln(n+10) - \ln(2n)$

$$a_n = \ln(n+10) - \ln(2n)$$
$$= \ln\left(\frac{n+10}{2n}\right)$$
$$= \ln\left(1/2 + 5/n\right)$$

$$\text{So} \lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(1/2 + 5/n\right) = \ln\left(\lim_{n\to\infty}\left(1/2 + 5/n\right)\right) = \boxed{\ln(1/2)}$$
$$= \boxed{-\ln 2}$$

either

2. Determine whether the following geometric series is convergent or divergent, and find the sum (in simplest form!) if it is convergent:

$$\sum_{n=1}^{\infty} \frac{(-8)^{n+1}}{9^n}$$

$$\sum_{n=1}^{\infty} \frac{(-8)^{n+1}}{9^n} = \sum_{n=1}^{\infty} \frac{(-8)^n \cdot -8}{9^n} = -8 \sum_{n=1}^{\infty} \left(-\frac{8}{9}\right)^n$$

This series is geometric with $r = -8/9$

$|r| = 8/9 < 1$

So the series converges. The sum is then...

$$-8 \cdot \frac{a}{1-r} = -8 \cdot \frac{-8/9}{1 - -8/9} = -8 \cdot \frac{-8/9}{\frac{9}{9} + \frac{8}{9}}$$

$$= -8 \cdot \frac{-8/9}{17/9}$$

$$= -8 \cdot -\frac{8}{\cancel{9}} \cdot \frac{\cancel{9}}{17}$$

$$= \boxed{64/17}$$

3. Determine which of the following series are absolutely convergent, which are conditionally convergent, and which are divergent.

(a) $\sum_{k=1}^{\infty} \frac{k(k+1)}{(k+3)^2}$

$$a_n = \frac{K(K+1)}{(K+3)^2} = \frac{K^2+K}{K^2+6K+9}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{K^2+K}{K^2+6K+9} = 1 \neq 0$$

Therefore, the series diverges by the Divergence Test.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+5}}$

$$a_n = \frac{(-1)^n n}{\sqrt{n^3+5}}$$

The series is alternating. For sufficiently large n, the sequence $|a_n|$ decreases and $\lim_{n\to\infty} a_n = 0$

Therefore, the series $\sum a_n$ converges by the Alternating Series Test.

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+5}}$$

$$> \sum_{n=1}^{\infty} \frac{n}{n^{3/2}} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}}_{\text{Diverges by p-test.}}$$

Therefore, $\sum |a_n|$ diverges by Comparison. So $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+5}}$ converges conditionally

(c) $\sum_{n=1}^{\infty} ne^{-n^2}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \frac{e^{n^2}}{e^{(n+1)^2}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{e^{n^2}}{e^{n^2+2n+1}} \right|$$

$$a_n = \frac{n}{e^{n^2}}$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{e^{n^2}}{e^{n^2} e^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{1}{e^{2n+1}} \right| = 0 < 1$$

Therefore, by the Ratio Test, the series converges absolutely.

4. Find the radius of convergence R and the interval of convergence I for the power series

$$\sum_{n=1}^{\infty} \frac{n(x+4)^n}{3^n}$$

$$\lim_{n\to\infty} \left| \frac{(n+1)(x+4)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n(x+4)^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} \cdot \frac{(x+4)^{n+1}}{(x+4)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{1}{3} \cdot x+4 \right|$$

$$= \left| \frac{x+4}{3} \right|$$

We want $\left| \frac{x+4}{3} \right| < 1 \rightleftarrows -3 < x+4 < 3$

$-7 < x < -1$

radius of convergence $\rightarrow R = \frac{-1 - -7}{2} = \frac{-1+7}{2} = \frac{6}{2} = 3$

If $x = -7 \ldots$

$$\sum_{n=1}^{\infty} n\frac{(-7+4)^n}{3^n} = \sum_{n=1}^{\infty} n\frac{(-3)^n}{3^n}$$

$$= \sum_{n=1}^{\infty} (-1)^n n$$

which diverges by the Divergence test

If $x = -1 \ldots$

$$\sum_{n=1}^{\infty} n\frac{(-1+4)^n}{3^n} = \sum_{n=1}^{\infty} n\frac{3^n}{3^n} = \sum_{n=1}^{\infty} n$$

which diverges by the Divergence Test.

Therefore, the radius of convergence is $R=3$ while the interval of convergence is $(-7,-1)$

In fact, we can even find the sum of

$$\sum_{n=1}^{\infty} \frac{n(x+4)^n}{3^n}$$

on the interval $(-7,-1)$. (See if you can justify the steps. There is only one critical step)

$$\sum_{n=1}^{\infty} \frac{n(x+4)^n}{3^n} = \sum_{n=1}^{\infty} n\left(\frac{x+4}{3}\right)^n = \frac{x+4}{3}\sum_{n=1}^{\infty} n\left(\frac{x+4}{3}\right)^{n-1}$$

$$= \frac{x+4}{3}\sum_{n=1}^{\infty} 3\frac{d}{dx}\left[\left(\frac{x+4}{3}\right)^n\right]$$

$$= x+4\sum_{n=1}^{\infty} \frac{d}{dx}\left[\left(\frac{x+4}{3}\right)^n\right]$$

$$= (x+4)\frac{d}{dx}\sum_{n=1}^{\infty}\left(\frac{x+4}{3}\right)^n$$

$$= (x+4)\frac{d}{dx}\left(\frac{\frac{x+4}{3}}{1-\frac{x+4}{3}}\right)$$

$$= (x+4)\frac{d}{dx}\left(-\frac{x+4}{x+1}\right)$$

$$= -(x+4)\frac{d}{dx}\left(\frac{x+4}{x+1}\right)$$

$$= -(x+4)\cdot\frac{-3}{(x+1)^2}$$

$$= \boxed{\frac{3(x+4)}{(x+1)^2}}$$

5. Find the Maclaurin series for the function $f(x) = \frac{1}{x+5}$

We know the Maclaurin series is....

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

0: $f(0) = \frac{1}{0+5} = \frac{1}{5}$

1: $f'(x) = \left.\frac{-1}{(x+5)^2}\right|_{x=0} = \frac{-1}{25} = \frac{-1}{5^2}$

2: $f''(x) = \left.\frac{2}{(x+5)^3}\right|_{x=0} = \frac{2}{5^3}$

$\vdots$

So $f^{(n)}(0) = \frac{(-1)^n n!}{5^{n+1}}$

Therefore, the Maclaurin series is...

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{\frac{(-1)^n n!}{5^{n+1}}}{n!} x^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^{n+1}}}$$

OR

Know the Maclaurin series for $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\frac{1}{x+5} = \frac{1}{5+x} = \frac{1}{5+x} \cdot \frac{1/5}{1/5} = \frac{1/5}{1+\left(\frac{x}{5}\right)} = \frac{1}{5} \cdot \frac{1}{1+\frac{x}{5}} = \frac{1}{5} \, \frac{1}{1-\left(\frac{-x}{5}\right)}$$

So using the above series...

$$\frac{1}{5} \cdot \frac{1}{1-\left(\frac{-x}{5}\right)} = \frac{1}{5} \cdot \sum_{n=0}^{\infty} \left(\frac{-x}{5}\right)^n = \frac{1}{5} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^{n+1}}}$$