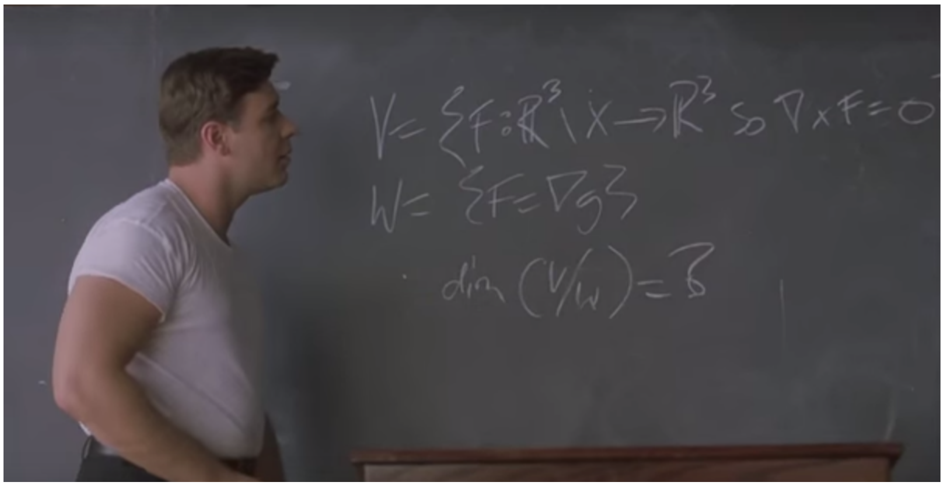
Beautiful Mind Problem

1 Introduction

In the Ron Howard film *A Beautiful Mind*, John Nash (played by Russell Crowe) gives his students a problem and says, "As I was saying, this problem here will take some of you many months to solve. For others among you, it will take you the term of your natural lives."



Nash's Problem:[1]

$$\begin{aligned} V &= \{F : \mathbb{R}^3 \setminus X \to \mathbb{R}^3 \colon \nabla \times F = \mathbf{0}\} \\ W &= \{F \colon F = \nabla g\} \\ \dim(V/W) &= 8 \end{aligned}$$

Let's clear the meaning of the notation a bit. The set V is the set of vector fields defined except on a set X such that they are curl free. The set W is the set of vector fields that are the gradient of some function, i.e. conservative vector fields. The $\dim(V/W)$ will be explained in the next section but is the question portion: find a set X such that there are 'only eight' vector fields which are curl free but not a gradient field. Despite Nash's threat, let's solve this problem in a time better measured in minutes or hours rather than in months or years.

In either case before giving a hint for the problem, a discussion of the notation is in order. The notation $A \setminus B$ is complement of B in A. Explicitly,

$$A \setminus B \overset{\text{def}}{=} \{a \in A \mid a \notin B\}$$

Essentially, take the elements of A and "throw out" any elements that also happen to be in B. For example, if $A = \{0, 1, 2, 3, 4, 5\}$ and $B = \{0, 4, 5\}$, then $A \setminus B = \{1, 2, 3\}$. You learn all the other symbols in the definition of V and W in the course.

[1] There are some that say what is written is $\dim(V/W) = ?$. This is an equally valid question – though more difficult. However, solving the problem assuming that it is $\dim(V/W) = 8$ is not only easier, it shows how to solve the case where $\dim(V/W) = ?$.

2 Linear Independence and Dimension

We say a collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent if and only if when $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, then $a_i = 0$ for all i. If a collection of vectors is not independent, we say that the collection of vectors is linearly dependent. For example, the collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \binom{1}{0}$ and $\mathbf{v}_2 = \binom{0}{1}$, is linearly independent as given

$$\begin{pmatrix}0\\0\end{pmatrix} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\begin{pmatrix}1\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a_1\\0\end{pmatrix} + \begin{pmatrix}0\\a_2\end{pmatrix} = \begin{pmatrix}a_1\\a_2\end{pmatrix}$$

Then $a_1 = a_2 = 0$. However, the collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{pmatrix}1\\0\\-2\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}2\\2\\-1\end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix}4\\6\\1\end{pmatrix}$, is linearly dependent as taking $a_1 = 2$, $a_2 = -3$, and $a_3 = 1$, we have

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = 2\begin{pmatrix}1\\0\\-2\end{pmatrix} + (-3)\begin{pmatrix}2\\2\\-1\end{pmatrix} + 1\begin{pmatrix}4\\6\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Now we say that a collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans (or generates) a space if every vector $\mathbf{v}$ can be written as

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

This is just a 'fancy' way of saying given any vector in our space (the space is 'probably' $\mathbb{R}^2$ or $\mathbb{R}^3$), we can find some linear combination of vectors from our collection that will give that vector. If the collection $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ not only spans the space but also is linearly independent, we say that the set is a basis for the space. The dimension of a space is the number of vectors in a basis. [It is not immediately obvious, but this number is unique and can be infinite.] For example, the space $\mathbb{R}^2$ has dimension 2 since given any vector $\binom{a_1}{a_2} \in \mathbb{R}^2$, we have

$$\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}a_1\\0\end{pmatrix} + \begin{pmatrix}0\\a_2\end{pmatrix} = a_1\begin{pmatrix}1\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\end{pmatrix}$$

Then the collection $\{\binom{1}{0}, \binom{0}{1}\}$ is a basis for $\mathbb{R}^2$ (we just showed that it spans $\mathbb{R}^2$ and we showed before that the vectors are independent). Therefore, $\dim \mathbb{R}^2 = 2$. Similarly, $\dim \mathbb{R}^3 = 3$ as $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^3$. Indeed, the dimension of $\mathbb{R}^n$ is n.

These spaces do not have to be a set of vectors; they could be a set of functions. Consider P_3 over the reals – the set of polynomials of at most degree 3; that is, P_3 is the set of polynomials of degree 0, degree 1, degree 2, and degree 3. Then the set $\{\mathbf{1}, x, x^2, x^3\}$ is clearly linearly independent. Furthermore, every polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ can be written as a combination of these 'vectors' so that $\dim P_3 = 4$. In general, $\dim P_n = n + 1$.

3 Nash's Problem

Returning to Nash's question, recall V is the set of vector fields defined everywhere in $\mathbb{R}^3$ (except perhaps on X) which are curl free and W is the set of gradient fields. The notation V/W means 'modding out by W'. This is more technical and we will not rigorously define what this means. Essentially, it means we treat the vector fields in V which are gradient fields as '0' (meaning the zero vector field, i.e. $\mathbf{F}(x, y, z) = \mathbf{0}$ for all x, y, z). We now have enough language to understand the problem. Nash's problem is to find a set of points X so that the vector fields (defined everywhere except maybe at the points of X) which are curl free but are *not* gradient fields has dimension 8, i.e. $\dim(V/W) = 8$. The fact that $\dim(V/W) = 8$ means there are eight vector fields $\{\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_8\}$ so that each of the **F**'s have the following properties:

(i) The **F**'s are in V: $\nabla \times \mathbf{F}_i = \mathbf{0}$ for any point $(x, y, z) \in \mathbb{R}^3$ that is not in X.

(ii) The set of $\mathbf{F}_i$'s are linearly independent: No sum of multiples of the **F**'s is a gradient field, i.e. is conservative. That is, there are no numbers $a_1, a_2, \dots, a_8$ and vector field F so that

$$\nabla \mathbf{F} = a_1\mathbf{F}_1 + a_2\mathbf{F}_2 + \cdots + a_8\mathbf{F}_8$$

(iii) The set of $\mathbf{F}_i$'s generate V/W: If **G** is a vector field with $\nabla \times \mathbf{G} = \mathbf{0}$, then there are numbers $a_1, a_2, \dots, a_8$ and a function field **F** so that

$$\nabla \mathbf{F} = \mathbf{G} - (a_1\mathbf{F}_1 + a_2\mathbf{F}_2 + \cdots + a_8\mathbf{F}_8)$$

I would recommend starting with the assuming that Crowe wrote an eight and solving that problem—it will help solve the general problem.

Example: If we did not throw out anything at all, that is $X = \emptyset$, then $\dim(V \setminus W) = 0$. Why?

Example: If we threw out just the origin, that is $X = \{\mathbf{0}\}$, then $\dim(V \setminus W) = 1$. I encourage you to think about this a bit.

Question: Look at the following function in 2-dimensions:

$$F(x, y) = \frac{-y}{x^2 + y^2}\,\hat{\mathbf{i}} + \frac{x}{x^2 + y^2}\,\hat{\mathbf{j}}$$

What is $\nabla \times F$? That is, what is curl F? What is the value of

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

where C is any circle (indeed any curve) about the origin? Is this a gradient field? Do you see any connection between the chosen loop, our choice of X, and the answer to the previous questions? How does this 2-dimensional problem help you find the set X such that $\dim(V \setminus W) = 8$?

Note that all of these problems show an intricate relationships between the 'geometry' of the space and functions on the space that are simple examples of de Rham cohomology. These are objects of interest in the fields Algebraic Topology and Differential Geometry – among others. However, Algebraic Topology and Differential Geometry are only a few of the many fields of Mathematics which study the relationship between function on a space and the 'geometry' of the space.

Bibliography

A Beautiful Mind. Ron Howard. Universal Pictures, 2002. Film.