Quiz 10 Calculus III Fall 2015

Solutions

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Solve the following problems. Each problem is worth 5 points.

Q1. Let $f(x,y) = ye^{x^2}$, and let C be the line segment from $(0,0)$ to $(1,2)$.

(a.) Evaluate the line integral $\int_C f(x,y)\,ds$.

$$r(t) = (1-t)\langle 0,0\rangle + t\langle 1,2\rangle = \langle 0,0\rangle + \langle t, 2t\rangle = \langle \underbrace{t}_{x}, \underbrace{2t}_{y}\rangle \text{ for } 0 \le t \le 1$$

$$r'(t) = \langle 1,2\rangle$$

$$|r'(t)| = \sqrt{1^2+2^2} = \sqrt{1+4} = \sqrt{5}$$

$$\int_C f(x,y)\,ds = \int_0^1 2t\, e^{t^2}\,dt \cdot \sqrt{5}$$

$u = t^2$

$du = 2t\,dt$

$dt = \frac{du}{2t}$

If $t=0 \to u=0$

If $t=1 \to u=1$

$$= \int_0^1 2t\, e^u \frac{du}{2t} \cdot \sqrt{5}$$

$$= \sqrt{5}\int_0^1 e^u\,du$$

$$\sqrt{5}e^u\Big|_0^1$$

$$\boxed{\sqrt{5}(e-1)}$$

(b.) Evaluate the line integral $\int_C f(x,y)\,dx$.

$x = t$

$dx = dt$

$$\int_C f\,dx = \int_0^1 2t\, e^{t^2}\,dt = e-1$$

By the above work

(c.) Evaluate the line integral $\int_C f(x,y)\,dy$.

$y = 2t$

$dy = 2dt$

$$\int_C f\,dy = \int_0^1 2t\, e^{t^2} \cdot 2dt = 2(e-1)$$

By the above work.

Q2. Let $\mathbf{F}(x,y) = xy^2\,\mathbf{i} + (x^2y + 2y)\,\mathbf{j}$.

(a.) Find the domain of $\mathbf{F}(x,y)$ and show that $\mathbf{F}(x,y)$ is a conservative vector field.

Domain: $\mathbb{R}^2$ which is open and simply connected.

The partial derivatives are continuous.

$$\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$$

$$2xy \stackrel{?}{=} 2xy \checkmark$$

Therefore, $\vec{F}$ is a conservative vector field.

(b.) Find the potential $g(x,y)$ of $\mathbf{F}(x,y)$.

$$F = \nabla f = \langle f_x, f_y \rangle = \langle xy^2, x^2y + 2y \rangle$$

$$f(x,y) = \int f_x\,dx = \int xy^2\,dx = \frac{x^2y^2}{2} + g(y)$$

$$f_y = x^2y + g'(y) = x^2y + 2y$$

$$g'(y) = 2y$$

$$\int g'(y)\,dy = \int 2y\,dy$$

$$g(y) = y^2 + K$$

$$\boxed{f(x,y) = \frac{x^2y^2}{2} + y^2 + K}$$

(b.) Use the potential $g(x,y)$ from part **(b.)** to compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the half circle in the right half space from $(0,1)$ to $(0,-1)$.

(Hint: Fundamental Theorem of Calculus for line integrals).

By the Fundamental Theorem for Line Integrals:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(0,-1) - f(0,1)$$

$$= \left(\frac{0^2(-1)^2}{2} + (-1)^2 + K\right) - \left(\frac{0^2(1)^2}{2} + 1^2 + K\right)$$

$$= (1+K) - (1+K)$$

$$\boxed{0}$$