Quiz 11 Calculus III Fall 2015

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Solutions

Solve the following problems. Each problem is worth 5 points.

Q1. Let $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$, and let C be the straight line segment from $(4, 6)$ to $(1, 0)$.

(b) Check that $\mathbf{F}$ is a **conservative** vector field.

Domain: $\mathbb{R}^2$, which is open and simply connected.

The partial derivatives are continuous.

$$\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$$

$$1 = 1 \checkmark$$

Therefore, $\vec{F}$ is a conservative vector field.

(b) Find a function $f(x, y)$ such that $\mathbf{F}(x, y) = \nabla f(x, y)$.

$$F = \nabla f = \langle f_x, f_y \rangle = \langle y, x \rangle$$

$$f(x,y) = \int f_x \, dx = \int y \, dx = xy + g(y)$$

$$f_y = x + g'(y) = x$$

$$g'(y) = 0$$

$$g(y) = K$$

$$\boxed{f(x,y) = xy + K}$$

(b) Use part (b) to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Hint: Fundamental Theorem for line integral).

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(1,0) - f(4,6)$$

$$= (1(0) + K) - (4(6) + K)$$

$$= \boxed{-24}$$

Via Fundamental Theorem for Line Integrals.

Q2.

1. Use Green's Theorem to compute

$$I = \int_C xy\, dx + x^2\, dy$$

where C is the boundary of the square with vertices $(0,0)$, $((1,0)$, $(1,1)$ and $(0,1)$ in the counterclockwise direction.

Assuming the above meets the conditions for Green's Theorem...

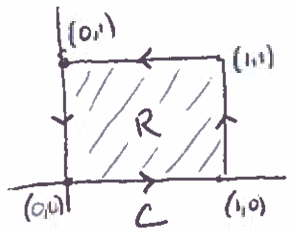
$$\oint_C M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$

$M = xy$

$N = x^2$

$\frac{\partial N}{\partial x} = 2x$

$\frac{\partial M}{\partial y} = x$



$$\oint_C xy\,dx + x^2 dy = \iint_R 2x - x\, dA$$

$$= \int_0^1\int_0^1 x\, dy\, dx$$

$$= \int_0^1 x\, dx \cdot \int_0^1 dy$$

$$= \frac{x^2}{2}\Big|_0^1 \cdot 1$$

$$= \boxed{1/2}$$

2. Use Green's Theorem to compute the **area** of the ellipse: $\frac{x^2}{4} + \frac{y^2}{9} = 1.$ $\Big\}$ $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

Area $= \iint 1\, dA$

So we want M, N such that

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$$

Here are two such choices:

We use $\rightarrow$ * $M = 0$, $N = x$

$M = -y$, $N = 0$

Of course, there are more choices.

Using Green's theorem (in Reverse)

$$\iint_R 1\, dA = \oint_C 0\, dx + x\, dy = \oint_C x\, dy$$

We need parametrize the ellipse. The easy way is

$r(t) = \langle 2\cos t, 3\sin t\rangle$ for $0 \le t \le 2\pi$

$x = 2\cos t$

$dy = 3\cos t$

$$\oint_C x\, dy = \int_0^{2\pi} 2\cos t \cdot 3\cos t = 6\int_0^{2\pi} \cos^2 t\, dt$$

$$6\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt = 3\int_0^{2\pi} 1 + \cos 2t\, dt$$

$$= 3\left(2\pi + \frac{\sin 2t}{2}\Big|_0^{2\pi}\right) = \boxed{6\pi}$$

We know area of ellipse is $\pi ab = \pi(2)3 = 6\pi$ $\longrightarrow$ This agrees with what we expected.