Group Quiz 9 Calculus III Fall 2015

Solutions

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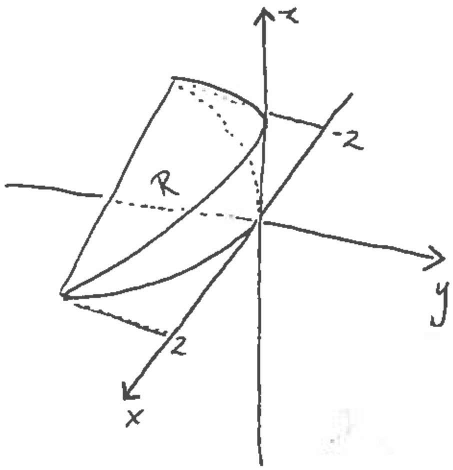
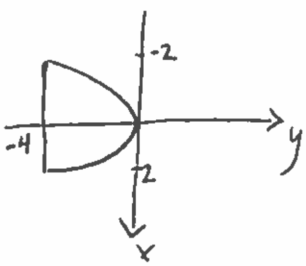
Solve the following problems. Each problem is worth 5 points.

Q1. Use a triple integral to find the volume of the solid enclosed by the cylinder $y = -x^2$ and the planes $z = 0$ and $z - y = 4$.





$$V = \iiint_R dV$$

$$= \int_{-2}^{2}\int_{-4}^{-x^2}\int_0^{y+4} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-4}^{-x^2} z\Big|_0^{y+4} dy\,dx$$

$$= \int_{-2}^{2}\int_{-4}^{-x^2} y+4 \; dy\,dx$$

$$= \int_{-2}^{2} \left(\frac{y^2}{2} + 4y\right)\Big|_{-4}^{-x^2} dx$$

$$= \int_{-2}^{2} \left(\frac{x^4}{2} - 4x^2\right) - (8-16)\, dx$$

$$= \int_{-2}^{2} \frac{x^4}{2} - 4x^2 + 8 \, dx$$

$$= \frac{1}{2}\int_{-2}^{2} x^4 - 8x^2 + 16 \, dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_{-2}^{2}$$

$$= \frac{1}{2}\left[\left(\frac{32}{5} - \frac{64}{3} + 32\right) - \left(-\frac{32}{5} - \frac{-64}{3} - 32\right)\right]$$

$$= \frac{1}{2}\left[\frac{32}{5} - \frac{64}{3} + 32 + \frac{32}{5} - \frac{64}{3} + 32\right]$$

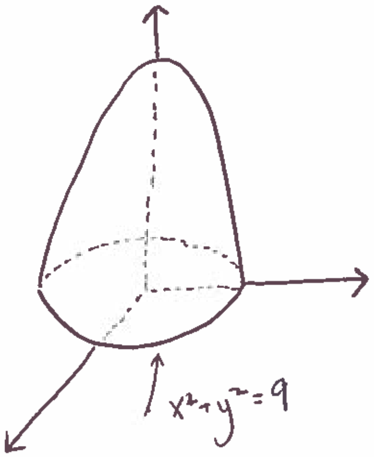
$$= \frac{1}{2}\left[\frac{64}{5} - \frac{128}{3} + 64\right]$$

$$= \frac{1}{2}\cdot\frac{512}{15} = \boxed{\frac{256}{15}}$$

Q2. (a.) Use cylindrical coordinates to evaluate

$$\iiint_E z\,dV$$

where E is the solid **in the first octant** that lies under the paraboloid $z = 9 - x^2 - y^2$.



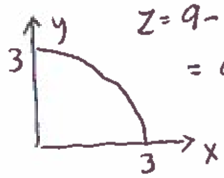
$$\iiint_E z\,dV = \int_0^{\frac{\pi}{2}}\int_0^3\int_0^{9-r^2} z \cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^3 r \int_0^{9-r^2} z\,dz\,dr$$

$$= \frac{\pi}{2}\int_0^3 r \cdot \frac{z^2}{2}\Big|_0^{9-r^2} dr$$

$$= \frac{\pi}{2}\int_0^3 r\,\frac{(9-r^2)^2}{2}\,dr$$

$$= \frac{\pi}{4}\int_0^3 r(9-r^2)^2\,dr$$



$z = 9-(x^2+y^2)$
$= 9-r^2$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$u = 9-r^2$
$du = -2r\,dr$
$dr = \frac{du}{-2r}$

$$= \frac{\pi}{4}\int_9^0 r\,u^2\,\frac{du}{-2r}$$

$$= \frac{\pi}{8}\int_0^9 u^2\,du$$

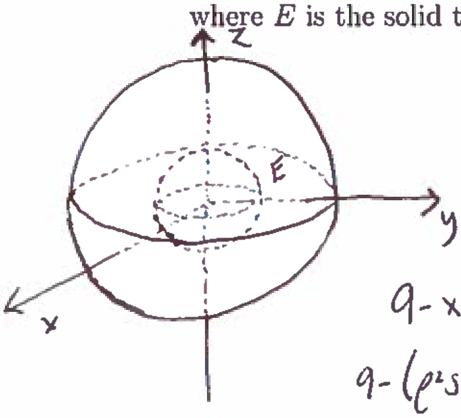
$$= \frac{\pi}{8}\,\frac{u^3}{3}\Big|_0^9 = \frac{\pi}{8}\cdot\frac{3^6}{3} = \frac{\pi}{8}3^5$$

$$= \frac{243\pi}{8}$$

(b.) Use spherical coordinates to evaluate

$$\iiint_E (9 - x^2 - y^2)\,dV$$

where E is the solid that lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 16$.



$$\iiint_E 9-x^2-y^2\,dV = \int_0^{\pi}\int_0^{2\pi}\int_2^4 (9-\rho^2\sin^2\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_2^4 9-\rho^2\sin\phi - \rho^4\sin^3\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\left(3\rho^3\sin\phi - \tfrac{1}{5}\rho^5\sin^3\phi\right)_2^4 d\theta\,d\phi$$

$$= \int_0^{2\pi} d\theta\int_0^{\pi}\left(3\rho^3\sin\phi - \tfrac{1}{5}\rho^5\sin^3\phi\right)_2^4 d\phi$$

$9 - x^2 - y^2$

$9 - (\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)$

$9 - \rho^2\sin^2\phi(\sin^2\theta + \cos^2\theta)$

$9 - \rho^2\sin^2\phi$

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$
$dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$\int \sin^3 x\,dx = \frac{\cos 3x - 9\cos x}{12}$

$$= 2\pi\int_0^{\pi} 168\sin\phi - 992\frac{\sin^3\phi}{5}\,d\phi$$

$$= 2\pi\left[\frac{-\cos\phi}{5} - \frac{248}{15}\cos(3\phi)\right]_0^{\pi}$$

$$= \frac{2144\pi}{15}$$