Summary of Convergence Tests for Series

Test	Series	Convergence or Divergence	Comments
n^{th} term test (or the zero test)	$\sum a_n$	Diverges if $\lim_{n\to\infty} a_n \neq 0$	Inconclusive if $\lim_{n\to\infty} a_n = 0$.
Geometric series	$\sum_{n=0}^{\infty} ax^n \left(\text{or} \sum_{n=1}^{\infty} ax^{n-1}\right)$	Converges to $\frac{a}{1-x}$ only if $\|x\| < 1$ Diverges if $\|x\| \geq 1$	Useful for comparison tests if the n^{th} term a_n of a series is similar to ax^n.
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges if $p > 1$ Diverges if $p \leq 1$	Useful for comparison tests if the n^{th} term a_n of a series is similar to $\frac{1}{n^p}$.
Integral	$\sum_{n=c}^{\infty} a_n \quad (c \geq 0)$ $a_n = f(n)$ for all n	Converges if $\int_c^{\infty} f(x)\,dx$ converges Diverges if $\int_c^{\infty} f(x)\,dx$ diverges	The function f obtained from $a_n = f(n)$ must be continuous, positive, decreasing and readily integrable for $x \geq c$.
Comparison	$\sum a_n$ and $\sum b_n$ with $0 \leq a_n \leq b_n$ for all n	$\sum b_n$ converges $\Longrightarrow \sum a_n$ converges $\sum a_n$ diverges $\Longrightarrow \sum b_n$ diverges	The comparison series $\sum b_n$ is often a geometric series or a p-series.
Limit Comparison*	$\sum a_n$ and $\sum b_n$ with $a_n, b_n > 0$ for all n and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$	$\sum b_n$ converges $\Longrightarrow \sum a_n$ converges $\sum b_n$ diverges $\Longrightarrow \sum a_n$ diverges	The comparison series $\sum b_n$ is often a geometric series or a p-series. To find b_n consider only the terms of a_n that have the greatest effect on the magnitude.
Ratio	$\sum a_n$ with $\lim_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} = L$	Converges (absolutely) if $L < 1$ Diverges if $L > 1$ or if L is infinite	Inconclusive if $L = 1$. Useful if a_n involves factorials or n^{th} powers.
Root*	$\sum a_n$ with $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = L$	Converges (absolutely) if $L < 1$ Diverges if $L > 1$ or if L is infinite	Test is inconclusive if $L = 1$. Useful if a_n involves n^{th} powers.
Absolute Value $\sum \|a_n\|$	$\sum a_n$	$\sum \|a_n\|$ converges $\Longrightarrow \sum a_n$ converges	Useful for series containing both positive and negative terms.
Alternating series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad (a_n > 0)$	Converges if $0 < a_{n+1} < a_n$ for all n and $\lim_{n\to\infty} a_n = 0$	Applicable only to series with alternating terms.

*The Root and Limit Comparison tests are not included in the current textbook used in Calculus classes at Bates College.

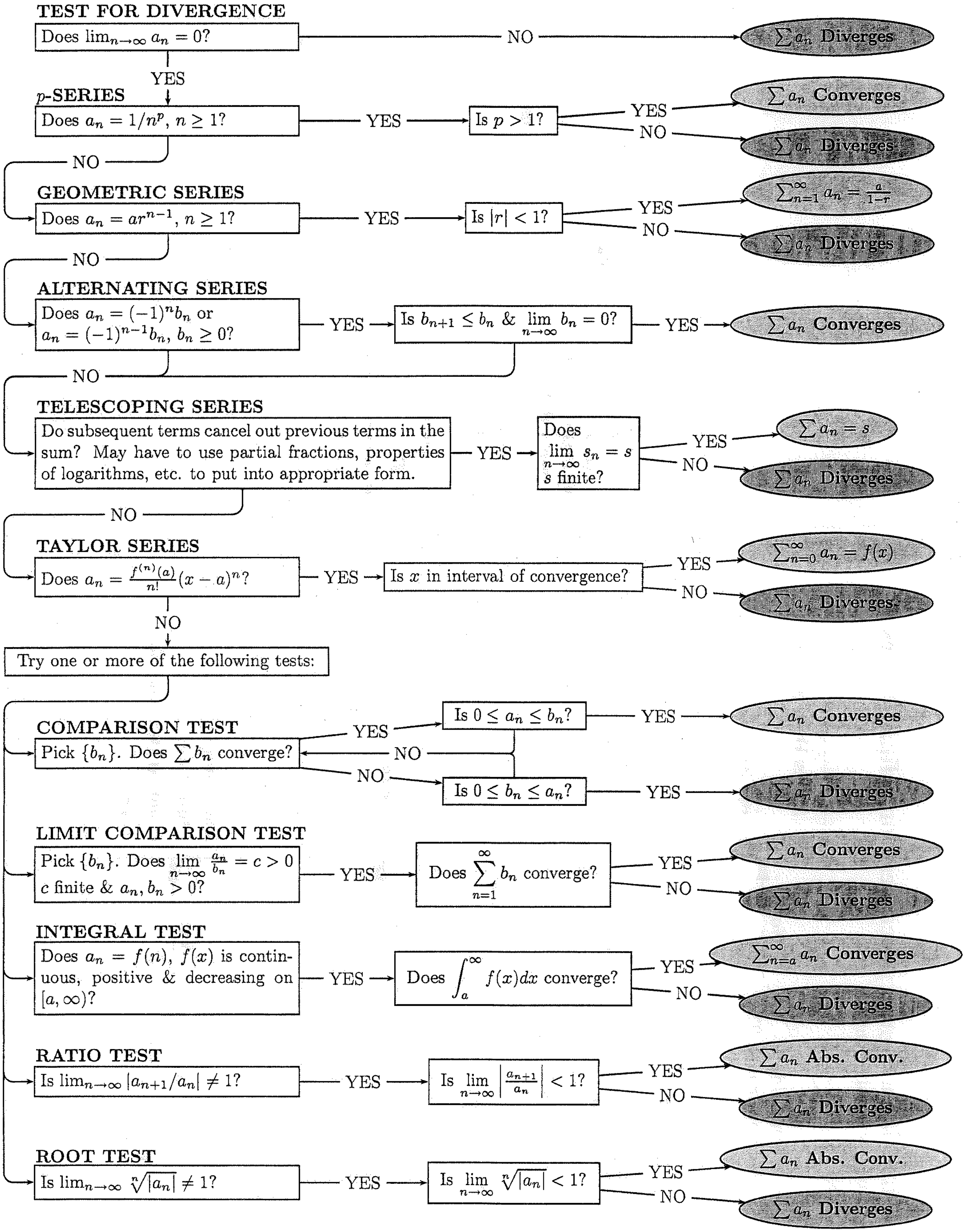
Strategy for Testing Series

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

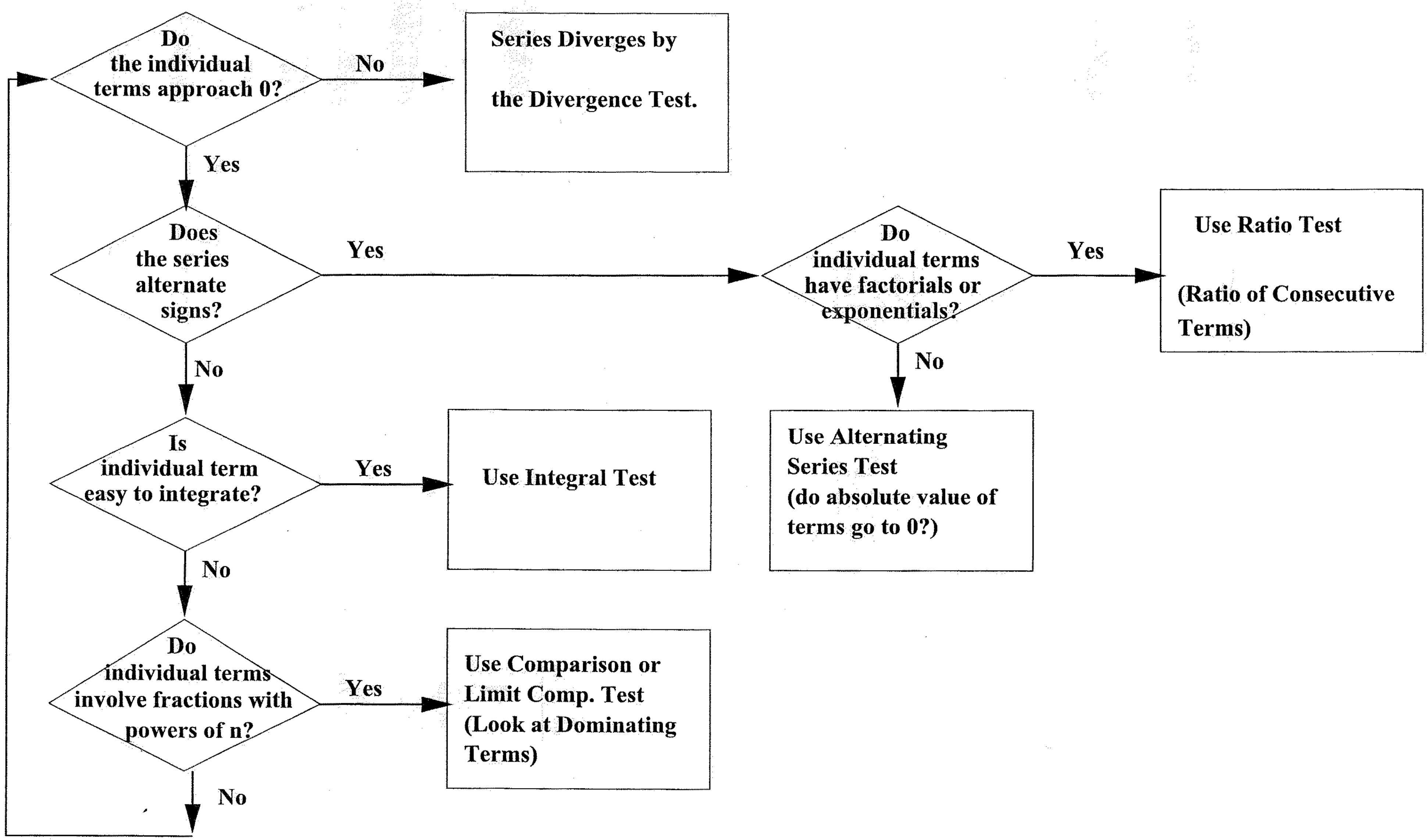
1. If the series is of the form $\Sigma\, 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.
2. If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. (The value of p should be chosen as in Section 8.3 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply the Comparison Test to $\Sigma\, |a_n|$ and test for absolute convergence.
4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\Sigma\, (-1)^{n-1}b_n$ or $\Sigma\, (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus, the Ratio Test should not be used for such series.
7. If $a_n = f(n)$, where $\int_1^\infty f(x)\, dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

Series Convergence/Divergence Flow Chart



Choosing a Convergence Test for Infinite Series

Courtesy David J. Manuel



Problems 1-38 from Stewart's *Calculus*, page 784

1. $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$

2. $\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$

5. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

6. $\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$

7. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

8. $\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$

9. $\sum_{k=1}^{\infty} k^2 e^{-k}$

10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

11. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln(n)}$

12. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+25}$

13. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

14. $\sum_{n=1}^{\infty} \sin(n)$

15. $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}$

16. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

17. $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

19. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$

20. $\sum_{k=1}^{\infty} \frac{k+5}{5^k}$

21. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

22. $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

23. $\sum_{n=1}^{\infty} \tan(1/n)$

24. $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$

25. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

26. $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

27. $\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+1)^3}$

28. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

29. $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n\sqrt{n}}$

30. $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$

31. $\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$

32. $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$

33. $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

34. $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2(n)}$

35. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

36. $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$

37. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)^n$

38. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)$