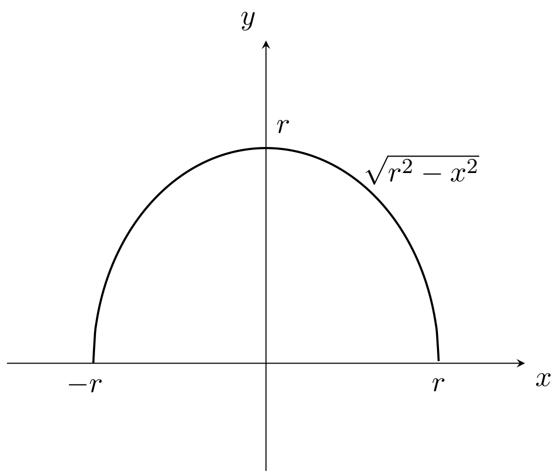
Volumes of Rotation — Solutions

Problem 1: The volume of a sphere of radius r is $V = \dfrac{4\pi}{3}r^3$. Prove this by finding the volume created by revolving the curve $y = \sqrt{r^2 - x^2}$ about the x-axis.

Solution.



Using disks, we have...

$$\begin{aligned}
V &= \pi \int R^2 - r^2 \, dx \\
&= \pi \int_{-r}^{r} \sqrt{r^2 - x^2}^{\,2} - 0^2 \, dx \\
&= \pi \int_{-r}^{r} (r^2 - x^2) \, dx \\
&= \pi \left(r^2 x - \frac{x^3}{3} \right) \Bigg|_{-r}^{r} \\
&= \pi \left(\left(r^2 - \frac{r^3}{3} \right) - \left(-r^3 - \frac{-r^3}{3} \right) \right) \\
&= \pi \left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3} \right) \\
&= \frac{4\pi}{3} r^3
\end{aligned}$$

Note that if $y = \sqrt{r^2 - x^2} \Leftrightarrow x = \pm\sqrt{r^2 - y^2}$. If we do this by shells, we have...

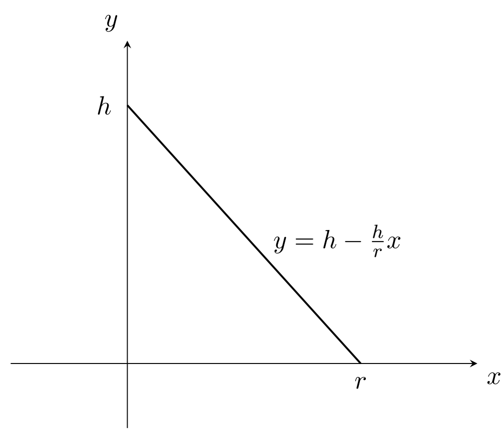
$$\begin{aligned}
V &= 2\pi \int r \cdot h \; dy \\
&= 2\pi \int_0^r y \cdot \left(\sqrt{r^2 - y^2} - (-\sqrt{r^2 - y^2}) \right) \, dy \\
&= 2\pi \int_0^r y \cdot 2\sqrt{r^2 - y^2} \; dy \\
&= 4\pi \int_0^r y \, \sqrt{r^2 - y^2} \; dy
\end{aligned}$$

Now let $u = r^2 - y^2$. Then we have $du = -2y \; dy$ so that $dy = \dfrac{du}{-2y}$. We need the new upper and lower limits: if $y = r$, then $u = 0$ and if $y = 0$ then $u = r^2$. Then we have...

$$\begin{aligned}
4\pi \int_0^r y\sqrt{r^2 - y^2} \; dy &= 4\pi \int_{r^2}^0 y\sqrt{u} \; \frac{du}{-2y} \\
&= -2\pi \int_{r^2}^0 \sqrt{u} \; du \\
&= 2\pi \int_0^{r^2} \sqrt{u} \; du \\
&= 2\pi \int_0^{r^2} u^{1/2} \; du \\
&= 2\pi \cdot u^{3/2} \cdot \frac{2}{3} \Big|_0^{r^2} \\
&= \frac{4\pi}{3} \, u^{3/2} \Big|_0^{r^2} \\
&= \frac{4\pi}{3} \, (r^2)^{3/2} - 0 \\
&= \frac{4\pi}{3} \, r^3
\end{aligned}$$

Problem 2: The volume of a right circular cone with base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. Prove this by finding the volume created by revolving the region bound by $y = h - \frac{h}{r}x$, x-axis, and y-axis about the y-axis.

Solution.



Note that $y = h - \frac{h}{r}x \Leftrightarrow x = \frac{r}{h}(y-h)$. Using disks, we have...

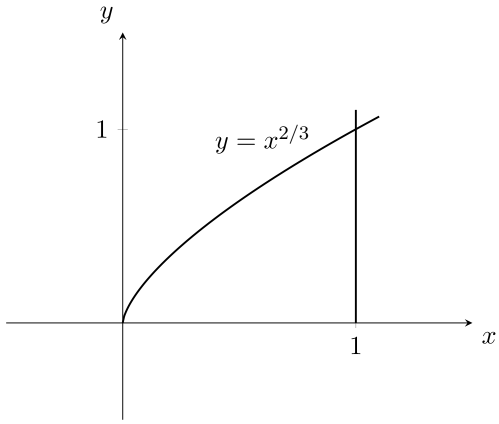
$$\begin{aligned}
V &= \pi \int (R^2 - r^2)\ dy \\
&= \pi \int_0^h \left(\left(\frac{r}{h}(y-h) \right)^2 - 0^2 \right)\ dy \\
&= \pi \int_0^h \frac{r^2}{h^2}\,(y-h)^2\ dy \\
&= \frac{\pi r^2}{h^2} \int_0^h (y^2 - 2yh + h^2)\ dy \\
&= \frac{\pi r^2}{h^2} \left[\frac{y^3}{3} - \frac{2hy^2}{2} + h^2 y \right]_0^h \\
&= \frac{\pi r^2}{h^2} \left[\left(\frac{h^3}{3} - \frac{2h^3}{2} + h^3 \right) - 0 \right]_0^h \\
&= \frac{\pi r^2}{h^2} \left(\frac{h^3}{3} - h^3 + h^3 \right) \\
&= \frac{\pi r^2}{h^2} \cdot \frac{h^3}{3} \\
&= \frac{\pi r^2 h}{3}
\end{aligned}$$

Now using shells, we have...

$$
\begin{aligned}
V &= 2\pi \int r \cdot h \; dx \\
&= 2\pi \int_0^r x \cdot \left(h - \frac{h}{r}x\right) \; dx \\
&= 2\pi \int_0^r \left(hx - \frac{h}{r}x^2\right) \; dx \\
&= 2\pi \left[\frac{hx^2}{2} - \frac{hx^3}{3r}\right]_0^r \\
&= 2\pi \left[\left(\frac{hr^2}{2} - \frac{hr^3}{3r}\right) - 0\right] \\
&= 2\pi \left(\frac{hr^2}{2} - \frac{hr^2}{3}\right) \\
&= 2\pi r^2 h \left(\frac{1}{2} - \frac{1}{3}\right) \\
&= 2\pi r^2 h \left(\frac{3}{6} - \frac{2}{6}\right) \\
&= 2\pi r^2 h \cdot \frac{1}{6} \\
&= \frac{\pi r^2 h}{3}
\end{aligned}
$$

Problem 3: Find the volume formed by the surface created by rotating the area bound by the curve $y = x^{2/3}$, $x = 1$, and the x-axis about the y-axis.

Solution.



Note that $y = x^{2/3} \Leftrightarrow x = y^{3/2}$. If we do this by washers, then we have...

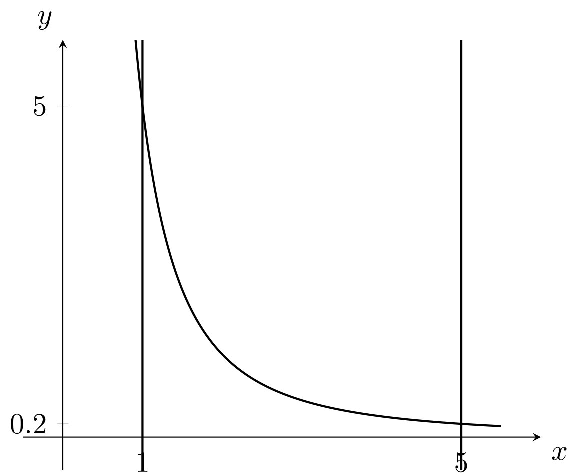
$$\begin{aligned}
V &= \pi \int R^2 - r^2 \, dy \\
&= \pi \int_0^1 \left(1^2 - (y^{3/2})^2\right) \, dy \\
&= \pi \int_0^1 \left(1 - y^3\right) \, dy \\
&= \pi \left(y - \frac{y}{4}\right) \Big|_0^1 \\
&= \pi \left(\left(1 - \frac{1}{4}\right) - \left(0 - 0\right)\right) \\
&= \pi \left(\frac{3}{4} - 0\right) \\
&= \frac{3\pi}{4}
\end{aligned}$$

However using shells, we have...

$$\begin{aligned}
V &= 2\pi \int r \cdot h \, dx \\
&= 2\pi \int_0^1 x \cdot x^{2/3} \, dx \\
&= 2\pi \int_0^1 x^{5/3} \, dx \\
&= 2\pi \cdot \frac{3}{8} x^{8/3} \Big|_0^1 \\
&= \frac{3\pi}{4} \left(1^{8/3} - 0\right) \\
&= \frac{3\pi}{4}
\end{aligned}$$

Problem 4: Find the volume of the figure formed by rotating the area $y = \dfrac{5}{x^2}$, $x = 5$, $x = 1$, and the x-axis around the y-axis.

Solution.



If we do this by shells:

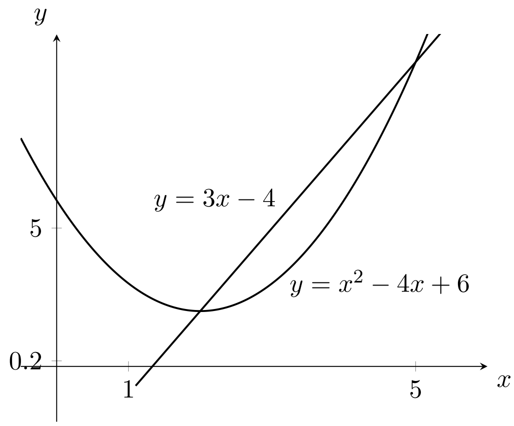
$$\begin{aligned}
V &= 2\pi \int r \cdot h \, dx \\
&= 2\pi \int_1^5 x \cdot \frac{5}{x^2} \, dx \\
&= 2\pi \int_1^5 \frac{5}{x} \, dx \\
&= 10\pi \int_1^5 \frac{1}{x} \, dx \\
&= 10\pi \ln|x| \, \Big|_1^5 \\
&= 10\pi \Big(\ln|5| - \ln|1| \Big) \\
&= 10\pi \ln(5)
\end{aligned}$$

Note that $y = \dfrac{5}{x^2} \Leftrightarrow x = \sqrt{\dfrac{5}{y}}$. Now if we do this by washers (notice we will need two integrals for this):

$$\begin{aligned}
V &= \pi \int R^2 - r^2 \; dy \\
&= \pi \int_{1/5}^{5} \left(\sqrt{\frac{5}{y}}^{\,2} - 1^2 \right) \; dy + \pi \int_{0}^{1/5} \left(5^2 - 1^2\right) \; dy \\
&= \pi \int_{1/5}^{5} \left(\frac{5}{y} - 1 \right) \; dy + \pi \int_{0}^{1/5} 24 \; dy \\
&= \pi \int_{1/5}^{5} \left(\frac{5}{y} - 1 \right) \; dy + 24\pi \int_{0}^{1/5} 1 \; dy \\
&= \pi \int_{1/5}^{5} \left(\frac{5}{y} - 1 \right) \; dy + 24\pi \cdot \frac{1}{5} \\
&= \frac{24\pi}{5} + \pi \int_{1/5}^{5} \left(\frac{5}{y} - 1 \right) \; dy \\
&= \frac{24\pi}{5} + \pi \left(5 \ln|y| - y \right) \Big|_{1/5}^{5} \\
&= \frac{24\pi}{5} + \pi \left(\left(5 \ln|5| - 5 \right) - \left(5 \ln|1/5| - \frac{1}{5} \right) \right) \\
&= \frac{24\pi}{5} + \pi \left(5 \ln(5) - 5 - 5 \ln|1/5| + \frac{1}{5} \right) \\
&= \frac{24\pi}{5} + \pi \left(5 \ln(5) - 5 - 5 \ln|5^{-1}| + \frac{1}{5} \right) \\
&= \frac{24\pi}{5} + \pi \left(5 \ln(5) - 5 - 5 \ln|1/5| + \frac{1}{5} \right) \\
&= \frac{24\pi}{5} + \pi \left(5 \ln(5) - 5 + 5 \ln(5) + \frac{1}{5} \right) \\
&= \frac{24\pi}{5} + \pi \left(10 \ln(5) - \frac{24}{5} \right) \\
&= \frac{24\pi}{5} + 10\pi \ln(5) - \frac{24\pi}{5} \\
&= 10\pi \ln(5)
\end{aligned}$$

Problem 5: Find the volume created by the surface formed by rotating the area between the curves $y = 3x - 4$ and $y = x^2 - 4x + 6$ around the line $y = -2$.

Solution.



First, we need to know where the curves intersect. Let $f(x) = 3x - 4$ and $g(x) = x^2 - 4x + 6$. Then

$$\begin{aligned} f(x) &= g(x) \\ 3x - 4 &= x^2 - 4x + 6 \\ x^2 - 7x + 10 &= 0 \\ (x-2)(x-5) &= 0 \end{aligned}$$

Therefore, they intersect when $x = 2$ and $x = 5$, which implies that they intersect at the points $(2, 2)$ and $(5, 11)$.

Using washers, we have…

$$\begin{aligned} V &= \pi \int (R^2 - r^2)\, dx \\ &= \pi \int_2^5 \Big((3x-4) - (-2)\Big)^2 - \Big((x^2 - 4x + 6) - (-2)\Big)^2\, dx \\ &= \pi \int_2^5 (3x-2)^2 - (x^2 - 4x + 8)^2\, dx \\ &= \pi \int_2^5 (9x^2 - 12x + 4) - (x^4 - 8x^3 + 32x^2 - 64x + 64)\, dx \\ &= \pi \int_2^5 -x^4 + 8x^3 - 23x^2 + 52x - 60\, dx \\ &= \pi \left(\frac{-x^5}{5} + 2x^4 - \frac{23x^3}{3} + 26x^2 - 60x \right) \Bigg|_2^5 \\ &= \pi \left(\left(\frac{-(5^5)}{5} + 2 \cdot 5^4 - \frac{23 \cdot 5^3}{3} + 26 \cdot 5^2 - 60 \cdot 5 \right) - \left(\frac{-(2^5)}{5} + 2 \cdot 2^4 - \frac{23 \cdot 2^3}{3} + 26 \cdot 2^2 - 60 \cdot 2 \right) \right) \\ &= \pi \left(\frac{50}{3} - \frac{-776}{15} \right) \\ &= \frac{342\pi}{5} \end{aligned}$$

Note that $y = 3x - 4 \Leftrightarrow x = \dfrac{y+4}{3}$ and $y = x^2 - 4x + 6 \Leftrightarrow x = 2 + \sqrt{y-2}$. [For the second equation, complete the square on the right side and then solve for x.] Now using shells, we have...

$$\begin{aligned}
V &= 2\pi \int r \cdot h \, dy \\
&= 2\pi \int_2^{11} \big(y - (-2)\big) \cdot \left((2 + \sqrt{y-2}\,) - \left(\frac{y+4}{3} \right) \right) \, dy \\
&= 2\pi \int_2^{11} (y+2) \cdot \left(2 + \sqrt{y-2} - \frac{1}{3}y - \frac{4}{3} \right) dy \\
&= 2\pi \int_2^{11} \left(\frac{4}{3} + 2\sqrt{y-2} + y\sqrt{y-2} - \frac{y^2}{3} \right) \, dy
\end{aligned}$$

We will not integrate this out. The first, second, and fourth terms are routine to integrate. For the third term, use u-substitution with $u = y - 2$, $du = dy$ to obtain $\int y\sqrt{y-2}\,dy = \int (u+2)\sqrt{u}\,du$, which is routine to integrate. When all the integration is done, we obtain...

$$\begin{aligned}
2\pi \int_2^{11} \left(\frac{4}{3} + 2\sqrt{y-2} + y\sqrt{y-2} - \frac{y^2}{3} \right) \, dy &= 2\pi \left(\frac{-16}{9} - \frac{56\sqrt{y-2}}{15} + \frac{4y}{3} + \frac{16}{15} y\sqrt{y-2} + \frac{2}{5} y^2 \sqrt{y-2} - \frac{y^3}{9} \right) \Bigg|_2^{11} \\
&= 2\pi \left(\frac{171}{5} - 0 \right) \\
&= \frac{342\pi}{5}
\end{aligned}$$