Name: *Caleb McWhorter — Solutions* **Quiz 8: Alternating Series & Ratio Test**
MAT 296 **Spring 2015**

Problem 1: Determine if the following alternating series diverges, converges conditionally, or converges absolutely.

$$\sum_{n=3}^{\infty} \frac{n}{n^2-5}$$

We know that $\lim_{n\to\infty} \frac{n}{n^2-5} = 0$. *Moreover, the sequence* $\left\{\frac{n}{n^2-5}\right\}$ *is decreasing as* $\frac{d}{dx}\left(\frac{x}{x^2-5}\right) = -\frac{x^2+5}{(x^2-5)^2} < 0$. *Then by the Alternating Series Test, this series converges conditionally. This series does not converge absolutely as* $\sum_{n=3}^{\infty} \frac{1}{n}$ *diverges by the p-test so that* $\sum_{n=3}^{\infty} \frac{1}{n} = \sum_{n=3}^{\infty} \frac{n}{n^2} < \sum_{n=3}^{\infty} \frac{n}{n^2-5}$ *diverges by the Comparison Test.*

Problem 2: Use the Ratio Test to determine if the following series diverges, converges conditionally, or converges absolutely.

$$\sum_{n=0}^{\infty} \frac{n\,4^n}{(2n)!}$$

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{(n+1)4^{n+1}}{(2(n+1))!}}{\frac{n4^n}{(2n)!}} \right| &= \lim_{n\to\infty} \frac{(n+1)4^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{n4^n} \\
&= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{4^{n+1}}{4^n} \cdot \frac{(2n)!}{(2n+2)!} \\
&= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{4^n\,4}{4^n} \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \\
&= \lim_{n\to\infty} \frac{n+1}{n} \cdot 4 \cdot \frac{1}{(2n+2)(2n+1)} \\
&= \lim_{n\to\infty} \frac{4(n+1)}{n(2n+1)(2n+2)} = 0 < 1
\end{aligned}$$

Then by the Ratio Test, $\sum_{n=0}^{\infty} \frac{n\,7^n}{(2n)!}$ *converges absolutely.*

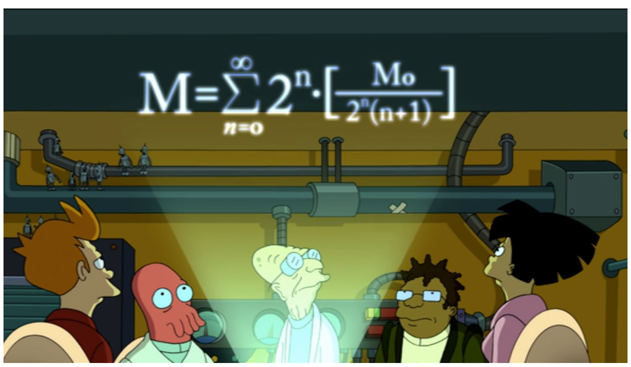
Bonus A: What is the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Bonus B: In the episode “Benderama” of the TV show Futurama™, the robot Bender avoids doing work by consuming some mass and making two smaller copies of himself, each 60% of his size, and makes the new copies do the work. Being equally lazy, the copies make copies of themselves to do their work and so forth. At one point in the episode, Professor Farnsworth runs into the room and announces that if this keeps up the total mass of Benders in the world will be given by

$$M = \sum_{n=0}^{\infty} 2^n \cdot \left[\frac{M_0}{2^n(n+1)} \right]$$

where M is the total mass of the Benders ansd M_0 is just some constant representing whatever the *original* Bender weighed. Everyone gasps in horror (except for Fry who does not understand). Explain why everyone is so worried. [Fun Fact: One of the creators/writers of the show has a PhD. in Math, which is why there are so many Math/Science jokes/references throughout the show. Fun Fact 2: The equation Professor Farnsworth gives and what he says are incorrect. *J'accuse!*]



$$M = \sum_{n=0}^{\infty} 2^n \cdot \frac{M_0}{2^n(n+1)} = \sum_{n=0}^{\infty} \frac{M_0}{n+1} = M_0 \sum_{n=0}^{\infty} \frac{1}{n+1} = M_0 \sum_{n=1}^{\infty} \frac{1}{n}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by the p-test (it is the Harmonic Series). But then the total mass of the Benders is infinite. In order for this to occur, the Benders will have to consume all the mass on Earth!*