Math 194: Exam 3 **Name:** *Caleb McWhorter — Solutions*
Summer – 2015
08/13/2015
140 Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains 12 pages (including this cover page) and 11 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	5	
2	9	
3	10	
4	12	
5	8	
6	6	
7	10	
8	9	
9	9	
10	10	
11	12	
Total:	100	

1. Complete the following parts:

   (a) (1 point) Define domain:

   The set of all possible inputs to a function, e.g. the "stuff" you can plug into the function.

   (b) (1 point) Define range:

   The set of all possible outputs to a function, e.g. the "stuff" you can get out of a function.

   (c) (1 point) Define linear function:

   A function whose graph is a straight line or a function of the form $f(\mathbf{x}) = \mathbf{b} + \mathbf{m}\mathbf{x}$.

   (d) (1 point) Define exponential function:

   A function of the form $f(x) = Ar^x$.

   (e) (1 point) Are the domains of $f(x) = \sqrt{x-1}$ and $g(x) = \log(x-1)$ the same? Explain.

   The domain of $f(x)$ *is* $x - 1 \geq 0 \implies x \geq 1$ *while the domain of* $g(x)$ *is* $x - 1 > 0 \implies x > 1$. *Therefore, the domains are not the same.*

2. Complete the following parts:

(a) (2 points) Find the equation of a line with slope 6 and y-intercept -5.

$y = mx + b$ so that $y = 6x - 5$

(b) (2 points) Find the equation of a line passing through the point $(3, 1)$ with slope 2.

$y = mx + b$ so that $y = 2x + b$. Using the point $(3, 1)$, we have $1 = 2(3) + b$ which implies $1 = 6 + b$ then $b = -5$. Therefore, $y = 2x - 5$.

(c) (2 points) Find the equation of a line passing through the points $(2, 7)$ and $(-1, -8)$.

We know $y = mx + b$. Now $m = \dfrac{7 - (-8)}{2 - (-1)} = \dfrac{15}{3} = 5$. Using the point $(2, 7)$, we have $7 = 5(2) + b$ so that $7 = 10 + b$ then $b = -3$. Now $y = 5x - 3$.

(d) (3 points) Find the equation of a parabola with x-intercepts $-2, 1$ and containing the point $(2, 12)$.

Because the x-intercepts are $x = -2, 1$, we know $y = a(x - (-2))(x - 1) = a(x + 2)(x - 1)$. Using the point $(2, 12)$, we have $12 = a(2 + 2)(2 - 1) = 4a$ so that $a = 3$. Then $y = 3(x + 2)(x - 1)$ or $y = 3x^2 + 3x - 6$.

3. Let $f(x) = 3 - 2x$ and $g(x) = 2x^2 + 7x - 15$

(a) (2 points) What is $f(2)$?

$f(2) = 3 - 2(2) = 3 - 4 = -1$

(b) (2 points) What is $f(g(x))$?

$f(g(x)) = f(2x^2 + 7x - 15) = 3 - 2(2x^2 + 7x - 15) = 3 - (4x^2 + 14x - 30) = -4x^2 - 14x + 33$

(c) (2 points) Factor $g(x)$.

$\underline{2}$ $\underline{15}$

$1 \cdot 2$ $1 \cdot 15$ — $1, 2$: $1, 30$; $2, 1$: $2, 15$

$3 \cdot 5$ — $1, 2$: $3, 10$ ✓; $2, 1$: $6, 5$

Therefore, $g(x) = (2x - 3)(x + 5)$.

(d) (4 points) Find the points of intersection between $f(x)$ and $g(x)$.

$$\begin{aligned} f(x) &= g(x) \\ 3 - 2x &= 2x^2 + 7x - 15 \\ 2x^2 + 9x - 18 &= 0 \\ (2x - 3)(x + 6) &= 0 \end{aligned}$$

Then $2x - 3 = 0$ *or* $x + 60$. *The first gives* $x = 3/2$ *and the second gives* $x = -6$. *Now* $f(3/2) = 3 - 2(3/2) = 3 - 3 = 0$ *and the second gives* $f(-6) = 3 - 2(-6) = 3 + 12 = 15$. *Therefore, the points of intersection are* $(3/2, 0)$ *and* $(-6, 15)$.

4. Macrosoft and Orange are both computer technologies companies. The profits of the companies, in hundreds of millions, t years after 2000 are given by $O(t) = 3t + 1$ and $M(t) = -t^2 + 5t + 6$, respectively.

   (a) (2 points) What is the x-intercept for $O(t)$? What might it represent in this context?

   We have $3t + 1 = 0$ so that $t = -1/3$ years. This could represent when the company started or when the company went bankrupt, etc.

   (b) (2 points) What is the y-intercept for $O(t)$? What does it represent in this context?

   $y = 1$ *hundred million. This represents the profit in the year 2000.*

   (c) (2 points) What is the maximum profit for $M(t)$?

   This is the turning point: $t = -b/(2a) = -5/(2\cdot -1) = 5/2$. *Then* $M(5/2) = 49/4 = 12.25$ *hundred million, aka $1,225,000,000.*

   (d) (6 points) What year do the companies make the same profit?

$$
\begin{aligned}
O(t) &= M(t) \\
3t + 1 &= -t^2 + 5t + 6 \\
-t^2 + 2t + 5 &= 0 \\
t &= \frac{-2 \pm \sqrt{(-2)^2 - 4(-1)5}}{2(-1)} \\
t &= \frac{-2 \pm \sqrt{4 + 20}}{-2} \\
t &= \frac{2 \pm \sqrt{24}}{2} \\
t &= \frac{2 \pm 2\sqrt{6}}{2} \\
t &= 1 \pm \sqrt{6}
\end{aligned}
$$

   $1 + 6\sqrt{6} \approx 3.449$ *and* $1 - \sqrt{6} \approx -1.449$. *Therefore, the companies make the same profit in 1998 and 2003.*

5. Complete the following parts:

(a) (2 points) Write the equation $5^x = 7$ in terms of logs.

$5^x = 7 \iff x = \log_5 7$

(b) (2 points) Write the equation $y = \log_6 2$ in terms of exponents.

$y = \log_6 2 \iff 6^y = 2$

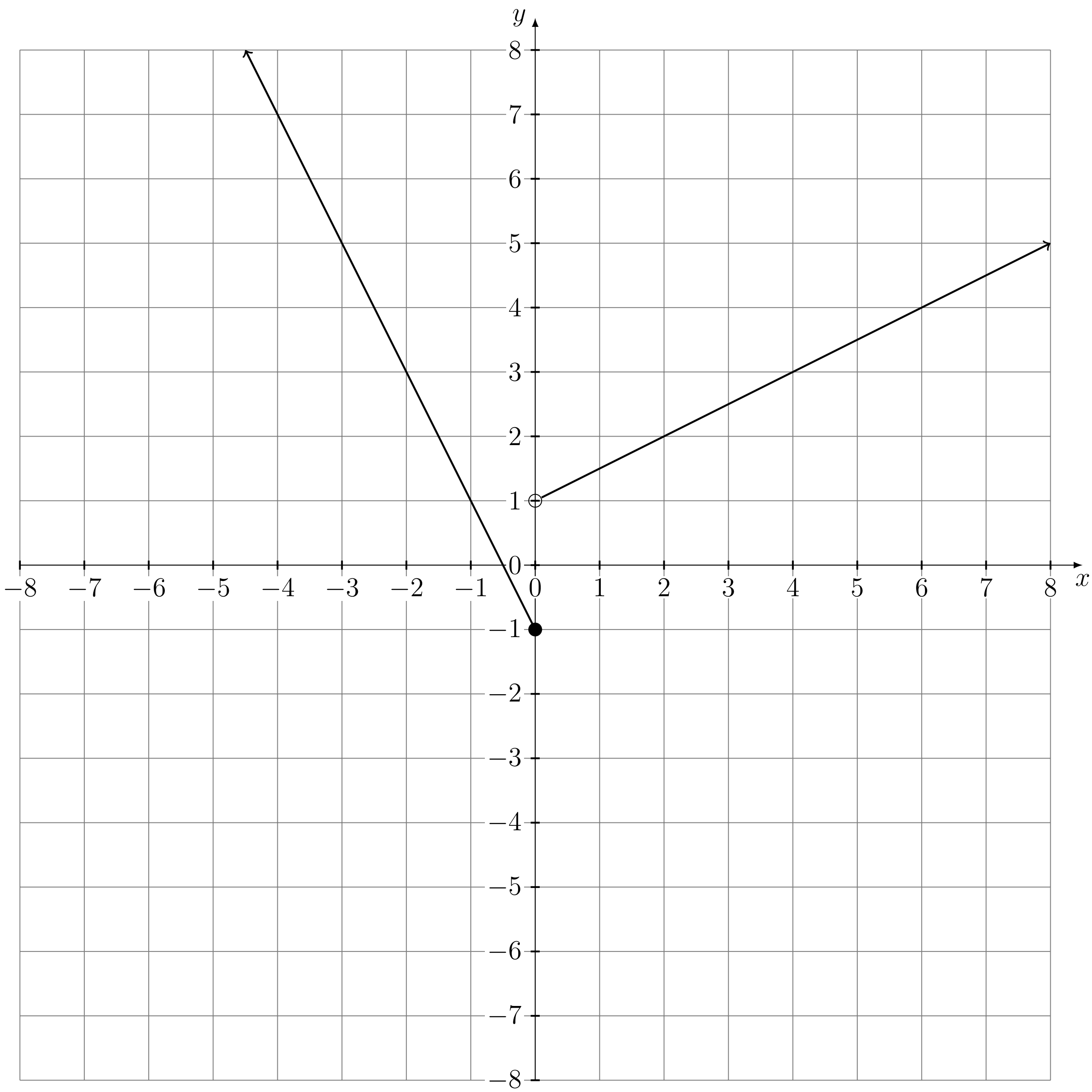
(c) (4 points) Find the equation of an exponential function containing the points $(-1, 45)$ and $(1, 5)$.

We have $y = ab^x$. *Bow using the point* $(-1, 45)$, *we have* $45 = ab^{-1} = \dfrac{a}{b}$ *so that* $a = 45b$. *Now using the point* $(1, 5)$, *we have* $5 = ab$. *Then* $5 = ab = (45b)b = 45b^2$. *Then* $b = 1/3$. *Then* $a = 45 \cdot 1/3 = 15$. *Therefore,*

$$y = 15\left(\frac{1}{3}\right)^x$$

6. (6 points) Use the coordinate plane below to graph the function

$$f(x) = \begin{cases} -2x - 1, & x \leq 0 \\ \frac{1}{2}x + 1, & x > 0 \end{cases}$$



7. Solve the following equations for x:

(a) (3 points)

$$4x^2 3^x - 3^{x+2} = 0$$

$$\begin{aligned} 4x^2 3^s - 3^x 3^2 &= 0 \\ 3^x(4x^2 - 9) &= 0 \\ 3^x(2x-3)(2x+3) &= 0 \end{aligned}$$

Therefore, $3^x = 0$ *or* $2x - 3 = 0$ *or* $2x + 3 = 0$*. The first yields no solution while the second give* $x = 3/2$ *and* $x = -3/2$*, respectively. Therefore,* $x = \pm 3/2$*.*

(b) (3 points)

$$\log_2(4x) - \log_3(x) = \log_3(x+2) + \log_2(2x)$$

$$\begin{aligned} \log_2(4x) - \log_2(2x) &= \log_3(x+2) + \log_3(x) \\ \log_2\left(\frac{4x}{2x}\right) &= \log_3(x(x+2)) \\ \log_2 2 &= \log_3(x(x+2)) \\ 1 &= \log_3(x(x+2)) \\ 3 &= x(x+2) \\ x^2 + 2x - 3 &= 0 \\ (x-1)(x+3) &= 0 \end{aligned}$$

Therefore, $x - 1 = 0$ *or* $x + 3 = 0$ *so that* $x = 1, -3$*. However,* $x = -3$ *leads to logs of negative numbers, which implies that* $x = -3$ *is an extraneous solution. Therefore,* $x = 1$*.*

(c) (4 points)

$$e^{2x} + 3e^x - 10 = 0$$

Let $y = e^x$*. Then*

$$\begin{aligned} y^2 + 3y - 10 &= 0 \\ (y-2)(y+5) &= 0 \end{aligned}$$

which implies $y - 2 = 0$ *or* $y + 5 = 0$*. But then* $y = 2 \iff e^x = 2 \iff x = \ln 2$ *or* $e^x = -5 \iff x = \ln(-5)$*, which is impossible. Then* $x = \ln 2$*.*

8. (9 points) Choose ***one*** of the following problems and complete it in its entirety. Place a checkmark next to the number of the problem you have chosen.

   (i) A bank account is opened with an initial deposit of \$4000. The account has a monthly compounded interest rate of 7%. Write a function, $M(t)$, representing the amount of money in the account t years after it was opened. How must is in the account after 3 years? When does the account have \$5500?

   (ii) A certain radioactive substance has a half-life time of 3.6 years. If 223kg of this substance is stored away, write a function, $A(t)$, representing the amount of this radioactive substance remaining after t years. How much of this substance is left after 2 years? When will only 127kg of this substance remain?

(i)

(a)

$$\begin{aligned} M(t) &= 4000\left(1+\frac{0.07}{12}\right)^{12t} \\ &= 4000(1.005833)^{12t} \\ &= 4000(1.07229)^{t} \end{aligned}$$

(b) $M(3) = \$4,931.70$

(c)

$$\begin{aligned} M(t) &= 5500 \\ 4000(1.07229)^t &= 5500 \\ 1.07229^t &= \frac{5500}{4000} \\ \ln(1.07229^t) &= \ln\left(\frac{5500}{4000}\right) \\ t\ln(1.07229) &= \ln\left(\frac{5500}{4000}\right) \\ t &= \frac{\ln\left(\frac{5500}{4000}\right)}{\ln(1.07229)} \\ t &\approx 4.56 \textit{ years} \end{aligned}$$

(ii)

(a) $A(t) = 223\left(\frac{1}{2}\right)^{t/3.6}$

(b) $A(2) = 151.728$ *kg*

(c)

$$\begin{aligned} A(t) &= 127 \\ 223\left(\frac{1}{2}\right)^{t/3.6} &= 127 \\ \left(\frac{1}{2}\right)^{t/3.6} &= \frac{127}{223} \\ \ln\left(\frac{1}{2}\right)^{t/3.6} &= \ln\left(\frac{127}{223}\right) \\ \frac{t}{3.6}\ln\left(\frac{1}{2}\right) &= \ln\left(\frac{127}{223}\right) \\ t &= 3.6\frac{\ln\left(\frac{127}{223}\right)}{\ln\left(\frac{1}{2}\right)} \\ t &\approx 2.924 \textit{ years} \end{aligned}$$

9. Complete the following parts:

(a) (3 points) How many digits does $163^{15,163}$ have?

$$\begin{aligned} &\log_{10} 163^{15163} \\ &15163 \log_{10} 163 \\ &33543.4006 \end{aligned}$$

Therefore, $163^{15,163}$ *has 33,544 digits.*

(b) (3 points) If $\ln(\log_3 x) = 13$, how many digits does x have?

$$\begin{aligned} \ln(\log_3 x) &= 13 \\ \log_3 x &= e^{13} \\ x &= 3^{e^{13}} \\ \log_{10} x &= \log_{10} 3^{e^{13}} \\ \log_{10} x &= e^{13} \log_{10} 3 \\ \log_{10} x &= 211084.832 \end{aligned}$$

Therefore, x *has* $211,085$ *digits.*

(c) (3 points) If you invest \$2000 in a bank account at 6% interest compounded continuously, how much money is in the account after 5 years?

$M(t)$ *is the money in the account after* t *years. Then* $M(t) = 2000e^{0.06t}$ *so that* $M(5) = 2000e^{0.06(5)} \approx \$2,699.72$.

10. Complete the following parts:

(a) (2 points) Find the equation of a line perpendicular to the line $y = \frac{3}{2}x + 2$ and containing the point $(6, 11)$.

We know $y = mx + b$ *so that* $y = -2/3x + b$. *Using the point* $(6, 11)$, *we have* $11 = -2/3(6) + b$ *so that* $11 = -4 + b$ *then* $b = 15$. *Then* $y = -2/3x + 15$.

(b) (2 points) If $f(x)$ is a function, what does the graph of $g(x) = f(x + 2) - 5$ look like compared to the graph of $f(x)$?

It looks like $f(x)$ *but shifted left 2 units and downward 5 units.*

(c) (2 points) What is the growth/shrink rate of the functions $r(s) = 7(1.15)^s$ and $q(n) = 6(0.96)^n$.

$r(s)$ *has a growth rate of 15% while* $q(n)$ *has a shrink rate of* 4%.

(d) (4 points) Simplify the following:

$$\ln(e^2) + \frac{\log_5\left(\frac{1}{25}\right)}{\log_3(9)} + e^{2\ln x} - \log_6(6^4)$$

$$2 + \frac{-2}{2} + e^{\ln x^2} - 4 = 2 - 1 + x^2 - 4 = x^2 - 3$$

11. Define $f(x)$ to be the following function

$$f(x) = \frac{(x-1)(x+2)(x+3)(x-4)}{(x-2)(x+5)(2x-12)(x-1)}$$

(a) (3 points) What are the roots of $f(x)$?

We have $(x-1)(x+2)(x+3)(x-4) = 0$ which imlpies $x = 1, -2, -3, 4$. However, the $x-1$ terms cancel so that the roots are $x = -2, -3, 4$.

(b) (3 points) What, if any, are the vertical asymptotes?

We have $(x-2)(x+5)(2x-12)(x-1) = 0$ so that $x = 2, -5, 6, 1$. But the $x-1$ terms cancel so that the vertical asymptotes are $x = 2, -5, 6$.

(c) (3 points) What, if any, are the horizontal asymptotes?

The degree of the numerator is the degree of the denominator. The leading coefficient of the numerator is 1 and for the denominator 2. The horizontal asymptote is therefore $y = 1/2$.

(d) (2 points) What, if any, are the holes of $f(x)$?

The $x-1$ term cancels so that

$$f(1) = \left.\frac{(x+2)(x+3)(x-4)}{(x-2)(x+5)(2x-12)}\right|_{x=1} = \frac{3 \cdot 4 \cdot -3}{-1 \cdot 6 \cdot -10} = \frac{-3}{5}$$

The hole is therefore $(1, -3/5)$.

(e) (1 point) Does $f(x)$ have a slant asymptote? Explain.

No, the degree of the numerator is not one more than the degree of the denominator.