Math 295: Exam 2
Fall – 2016
10/21/2016
55 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 8 pages (including this cover page) and 7 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	15	
2	10	
3	15	
4	10	
5	15	
6	20	
7	15	
Total:	100	

1. (15 points) Evaluate the following derivatives—you do *not* need to simplify:

$$\frac{d}{dx}\left(\tan^2 x\left(x^5+\pi-\frac{2}{\sqrt[3]{x}}\right)\log_5 x\right)=$$

$$2\tan x\sec^2 x\left(x^5+\pi-\frac{2}{\sqrt[3]{x}}\right)\log_5 x+\tan^2 x\left(5x^4+\frac{2}{3\sqrt[3]{x^4}}\right)\log_5 x+\tan^2 x\left(x^5+\pi-\frac{2}{\sqrt[3]{x}}\right)\frac{1}{x\ln 5}$$

$$\frac{d}{dx}\left(\frac{x^2e^{5x^3}}{\sin(1-x)}\right)=$$

$$\frac{\sin(1-x)(2xe^{5x^3}+x^2e^{5x^3}15x^2)-\cos(1-x)\cdot -1\cdot x^2e^{5x^3}}{\sin^2(1-x)}$$

$$\frac{d}{dx}\,2^{\ln(\arctan(x^3))}=$$

$$2^{\ln\arctan x^5}\ln 2\cdot\frac{1}{\arctan x^3}\cdot\frac{1}{1+(x^3)^2}\cdot 3x^2$$

2. (10 points) Find the derivative of $x^{\sec x}$.

Let $y = x^{\sec x}$. Then...

$$y = x^{\sec x}$$

$$\ln y = \ln x^{\sec x}$$

$$\ln y = \sec x \ln x$$

$$\frac{d}{dx} \ln y = \frac{d}{dx} \sec x \ln x$$

$$\frac{1}{y} y' = \sec x \tan x \ln x + \frac{\sec x}{x}$$

$$\frac{dy}{dx} = y \left(\sec x \tan x \ln x + \frac{\sec x}{x} \right)$$

$$\frac{dy}{dx} = x^{\sec x} \left(\sec x \tan x \ln x + \frac{\sec x}{x} \right)$$

Or using the rule $\frac{d}{dx}[f(x)]^{g(x)} = g(x)[f(x)]^{g(x)-1} \cdot f'(x) + [f(x)]^{g(x)} \ln(f(x)) \cdot g'(x)$, we have

$$\frac{d}{dx} x^{\sec x} = (\sec x) x^{\sec(x)-1} \cdot 1 + x^{\sec x} \ln x \cdot \sec x \tan x$$

3. (15 points) Given the fact that $y^2 - xy = 5$, find $\dfrac{d^2y}{dx^2}$ without explicitly solving for y.

$$\begin{aligned}
y^2 - xy &= 5 \\
\frac{d}{dx}\left(y^2 - xy\right) &= \frac{d}{dx}5 \\
2yy' - (y + xy') &= 0 \\
2yy' - y - xy' &= 0 \\
2yy' - xy' &= y \\
y'(2y - x) &= y \\
\frac{dy}{dx} &= \frac{y}{2y - x}
\end{aligned}$$

But then we have...

$$\begin{aligned}
\frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}\left(\frac{y}{2y-x}\right) \\
\frac{d^2y}{dx^2} &= \frac{(2y-x)y' - (2y'-1)y}{(2y-x)^2} \\
\frac{d^2y}{dx^2} &= \frac{(2y-x)\dfrac{y}{2y-x} - \left(2\dfrac{y}{2y-x} - 1\right)y}{(2y-x)^2} \\
\frac{d^2y}{dx^2} &= \frac{y - \left(\dfrac{2y}{2y-x} - \dfrac{2y-x}{2y-x}\right)y}{(2y-x)^2} \\
\frac{d^2y}{dx^2} &= \frac{y - \dfrac{xy}{2y-x}}{(2y-x)^2} \\
\frac{d^2y}{dx^2} &= \frac{\dfrac{2y^2-xy}{2y-x} - \dfrac{xy}{2y-x}}{(2y-x)^2} \\
\frac{d^2y}{dx^2} &= \frac{\dfrac{2y^2-2xy}{2y-x}}{(2y-x)^2} \\
\frac{d^2y}{dx^2} &= \frac{2y(y-x)}{(2y-x)^3}
\end{aligned}$$

4. (10 points) Prove the fact that $\dfrac{d}{dx}\operatorname{arcsec} x = \dfrac{1}{|x|\sqrt{x^2-1}}$.

Let $y = \operatorname{arcsec} x$. *Then* $\sec y = x$ *so that*

$$\begin{aligned}\frac{d}{dx}\sec y &= \frac{d}{dx}x\\ y'\sec y\tan y &= 1\\ \frac{dy}{dx} &= \frac{1}{\sec y\tan y}\end{aligned}$$

But recall that $\sin^2 y + \cos^2 y = 1$. *Then* $\dfrac{\sin^2}{\cos^2 y} + \dfrac{\cos^2 y}{\cos^2 y} = \dfrac{1}{\cos^2 y}$ *so that* $\tan^2 y + 1 = \sec^2 y$.

Therefore, $\tan y = \pm\sqrt{\sec^2 y - 1}$. *By the work above,* $\sec y = x$ *so that* $x^2 = \sec^2 y$. *Then* $\tan y = \sqrt{x^2-1}$ *and* $\sec y = x$.

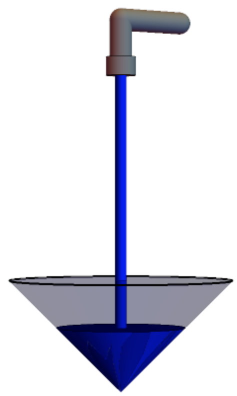
$$\frac{dy}{dx} = \frac{1}{\sec y\tan y} = \frac{1}{x\cdot(\pm\sqrt{x^2-1})}$$

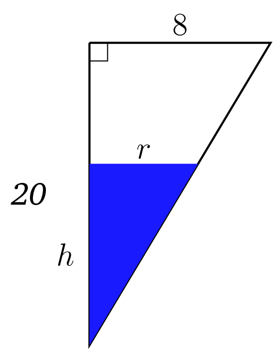
To determine the sign for this derivative, note that $\dfrac{dy}{dx}$ *has the same sign as* $\dfrac{1}{\sec y\tan y} =$

$\dfrac{\cos^2 y}{\sin y}$. *But* $\cos^2 y \geq 0$ *so that* $\dfrac{dy}{dx}$ *has the same sign as* $\sin y$. *If* $0 < \operatorname{arcsec} x < \dfrac{\pi}{2}$, *then* $x > 0$ *and if* $\dfrac{\pi}{2} < \operatorname{arcsec} x < \pi$, *then* $x < 0$. *Therefore,*

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{|x|\sqrt{x^2-1}}\\ \frac{d}{dx}\operatorname{arcsec} x &= \frac{1}{|x|\sqrt{x^2-1}}\end{aligned}$$

5. (15 points) A large conical water tower has a height of 20 ft and a radius of 8 ft. Water is being pumped into the tower for storage at a rate of 100 cubic ft per minute, as shown below. How quickly is the water level rising when the tower is filled to a depth of 5 ft? *We have* $V = \frac{1}{3}\pi r^2 h$. *Examine a cross section:*





By similar triangles,

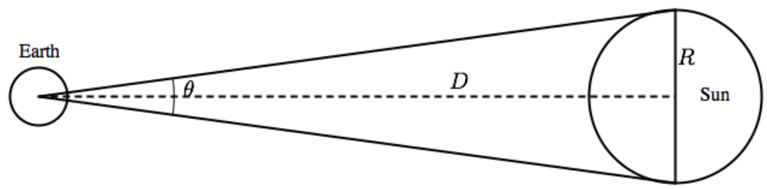
$$\frac{r}{h} = \frac{8}{20}$$
$$\frac{r}{h} = \frac{2}{5}$$
$$r = \frac{2}{5}h$$

Therefore, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{1}{3}\pi\frac{4}{25}h^3$. *Then*

$$\frac{dV}{dt} = \frac{d}{dt}V = \frac{d}{dt}\left(\frac{1}{3}\pi \cdot \frac{4}{25}h^3\right) = \frac{4\pi h^2}{25} \cdot \frac{dh}{dt}$$

Therefore, $\frac{dh}{dt} = \frac{25\frac{dV}{dt}}{4\pi h^2}$. *We know that* $h = 5$ *ft and* $\frac{dV}{dt} = 100$ *ft*^{3}*/min. Then*

$$\frac{dh}{dt} = \frac{25 \cdot 100}{4\pi(5)^2} = \frac{100}{4\pi} = \frac{25}{\pi} \approx 7.96 \text{ ft/min}$$



6. Consider the diagram above, where D is the distance to the Sun in meters and R is the radius of the Sun in meters. Derive a method of approximating the radius of the Sun as follows:

(a) (5 points) Find the linearization of $f(\theta) = \sin\theta$ at $\theta = 0$. Use this show that $\sin\theta \approx \theta$ when θ is close to 0, i.e. when θ is "small."

Let $f(\theta) = \sin\theta$. Then $f'(\theta) = \cos\theta$. We have $f(0) = \sin 0 = 0$ and $f'(0) = \cos 0 = 1$. Then $\ell(\theta) = 0 + 1(\theta - 0) = \theta$. If θ is 'small', i.e. 'close' to 0, $f(\theta) \approx \ell(\theta)$. But then $\sin\theta \approx \theta$ for $\theta \approx 0$.

(b) (5 points) Use the previous part to show that $\tan\theta \approx \theta$ when θ is "small", i.e. $\theta \approx 0$.

If θ is 'small', then $\sin\theta \approx \theta$ by (a). But if $\theta \approx 0$, then $\cos\theta \approx \cos 0 = 1$. But then $\tan\theta = \dfrac{\sin\theta}{\cos\theta} \approx \dfrac{\theta}{1} = \theta$. But then $\tan\theta \approx \theta$ when θ is 'small."

(c) (5 points) Use right triangle trigonometry to find an equation relating θ, D, and R in the diagram above.

We have $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{R}{D}$.

(d) (5 points) Light travels at constant velocity c. Suppose that the angle θ is "small." Combine the previous two parts to show $R \approx \dfrac{\theta ct}{2}$, where t is the time it takes light to travel from the Sun to the Earth. [Recall $D = vt$, where t is time and v is velocity.]

If θ is 'small', then $\theta/2$ is 'small. We know from (b) that $\tan\theta \approx \theta$ when θ is small. But then $\tan\left(\dfrac{\theta}{2}\right) \approx \dfrac{\theta}{2}$. Therefore, $\dfrac{\theta}{2} \approx \tan\left(\dfrac{\theta}{2}\right) = \dfrac{R}{D}$. Then $\dfrac{\theta}{2} \approx \dfrac{R}{D}$ so that $R \approx \dfrac{D\theta}{2}$. But $D = vt = ct$. Then

$$R \approx \frac{ct\theta}{2}$$

7. (15 points) In Physics for reversible adiabatic processes involving "nice" gases,

$$TV^{\gamma-1} = C,$$

where T is the temperature of the gas in Kelvin, V is the volume of the gas, γ is the ratio of specific heats at constant pressure, and C is a constant depending on the gas involved. This can be applied to rising helium balloons in the atmosphere, where in this case $\gamma \approx 1.66$.

(a) Use the equation $TV^{\gamma-1} = C$ to find the equation of the rate of change of temperature for the gas in the balloon as it rises into the atmosphere.

$$\begin{aligned}
TV^{\gamma-1} &= C \\
\frac{d}{dt}\,TV^{\gamma-1} &= \frac{d}{dt}\,C \\
\frac{dT}{dt}\,V^{\gamma-1} + T(\gamma-1)V^{\gamma-2}\,\frac{dV}{dt} &= 0 \\
V^{\gamma-1}\frac{dT}{dt} &= -T(\gamma-1)V^{\gamma-2}\frac{dV}{dt} \\
\frac{dT}{dt} &= \frac{-T(\gamma-1)V^{\gamma-2}\frac{dV}{dt}}{V^{\gamma-1}} \\
\frac{dT}{dt} &= -\frac{T(\gamma-1)\frac{dV}{dt}}{V}
\end{aligned}$$

(b) Use the equation from (a), the fact that volume and temperature (in Kelvin) are positive quantities, and the fact that the rising air expands in time to determine whether the temperature of the gas is increasing or decreasing.

We know $T > 0$, $V > 0$, and $\gamma \approx 1.66$. Then $\gamma - 1 > 0$. The balloon is rising and the helium expanding so $\frac{dV}{dt} > 0$. Then $T(\gamma-1)\frac{dV}{dt} > 0$ and $V > 0$. But then $\frac{T(\gamma-1)\frac{dV}{dt}}{V} > 0$. Then

$$\frac{dT}{dt} = -\frac{T(\gamma-1)\frac{dV}{dt}}{V} < 0$$

Therefore, the temperature of the helium is decreasing in time.