Math 295: Exam 3 **Name:** *Caleb McWhorter — Solutions*
Fall – 2016
11/18/2016
55 Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains 7 pages (including this cover page) and 5 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	15	
2	15	
3	10	
4	10	
5	10	
Total:	60	

1. (15 points) Compute each of the following limits. Be sure to justify your answer completely.

(a) $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3} \overset{L.H.}{=} \lim\limits_{x\to 0} \dfrac{1-\cos x}{3x^2} \overset{L.H.}{=} \lim\limits_{x\to 0} \dfrac{\sin x}{6x} \ln \lim\limits_{x\to 0} \dfrac{\cos x}{6} = \dfrac{1}{6}$

(b) $\lim\limits_{x\to\infty} \dfrac{x^2+3x+1}{e^{2x}-x+2} \overset{L.H.}{=} \lim\limits_{x\to\infty} \dfrac{2x+3}{2e^{2x}-1} = \overset{L.H.}{=} \lim\limits_{x\to\infty} \dfrac{2}{4e^{2x}} = 0$

(c) $\lim\limits_{x\to\frac{\pi}{2}} (\sec x - \tan x) = \lim\limits_{x\to\frac{\pi}{2}} \left(\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}\right) = \lim\limits_{x\to\frac{\pi}{2}} \dfrac{1-\sin x}{\cos x} \overset{L.H.}{=} \lim\limits_{x\to\frac{\pi}{2}} \dfrac{-\cos x}{-\sin x} = 0$

(c) $\lim\limits_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{1/x^2} \overset{L.H.}{=} \lim_{x\to 0^+} \dfrac{1/x}{-2/x^3} = \lim_{x\to 0^+} \dfrac{x^3}{-2x} = \lim_{x\to 0^+} \dfrac{x^2}{-2} = 0$

(d) $\lim\limits_{x\to 0} (\cos x)^{3/x^2}$

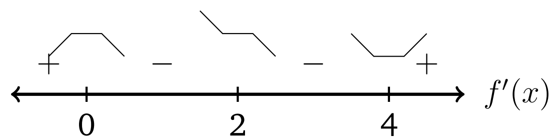
$$y = \lim_{x\to 0}(\cos x)^{3/x^2}$$
$$\ln y = \ln\left(\lim_{x\to 0}(\cos x)^{3/x^2}\right)$$
$$\ln y = \lim_{x\to 0} \ln(\cos x)^{3/x^2}$$
$$\ln y = \lim_{x\to 0} \frac{3}{x^2}\ \ln \cos x$$
$$\ln y = \lim_{x\to 0} \frac{3 \ln \cos x}{x^2}$$
$$\ln y \overset{L.H.}{=} \lim_{x\to 0} \frac{3\frac{-\sin x}{\cos x}}{2x}$$
$$\ln y = \lim_{x\to 0} \frac{-3 \tan x}{2x}$$
$$\ln y = \lim_{x\to 0} \frac{-3 \sec^2 x}{2}$$
$$\ln y = \frac{-3}{2}$$
$$y = e^{-3/2} = \frac{1}{\sqrt{e^3}}$$

2. (15 points) Let $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$.

(a) Find and classify any critical values for $f(x)$.

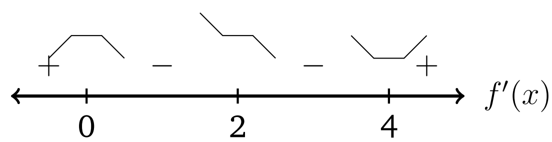
$$f'(x) = \frac{(x-2)(2x-2) - 1(x^2 - 2x + 4)}{(x-2)^2} = \frac{2x^2 - 2x - 4x + 4 - x^2 + 2x - 4}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}$$

Therefore, $f'(x) = 0$ when $x = 0$ or $x = 4$. Furthermore, $f'(x)$ is undefined at $x = 2$. Using the First Derivative Test,



Therefore, $x = 0$ is a maximum, $x = 4$ is a minimum, and $x = 2$ is neither (the function is not even defined there).

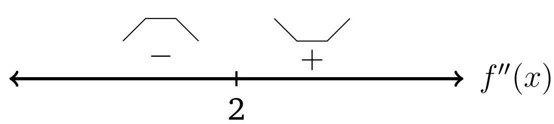
(b) Find the intervals on which $f(x)$ is increasing and decreasing.



We know $f(x)$ is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$.. Using (a), we know that $f(x)$ is increasing on $(-\infty, 0) \cup (4, \infty)$ and decreasing on $(0, 2) \cup (2, 4)$.

(c) Find the intervals on which $f(x)$ is concave and convex.

$$f''(x) = \frac{(x-2)^2(2x-4) - 2(x-2)(x^2-4x)}{(x-2)^4} = \frac{2\big((x-2)^2 - (x^2-4x)\big)}{(x-2)^4} = \frac{8}{(x-2)^3}$$



Therefore, $f(x)$ is concave on $(-\infty, 2)$ and convex on $(2, \infty)$. Note $x = 2$ would be a point of inflection for $f(x)$ if $f(x)$ were in the domain for $f(x)$.

(d) Find the local and absolute maxima/minima on $[0, 6]$.

We know $f(0) = -2$, $f(6) = 7$, and $f(4) = 6$. By the work above, there is a local max at $x = 0$ and a local minimum at $x = 4$. Examining the infinite discontinuity at $x = 2$ and using $x^2 - 2x + 4 > 0$ at $x = 2$, we know there is no absolute minimum or maximum.

3. (10 points) Complete the following parts:

(a) If $0 < a < b$, show that
$$\sqrt{b} - \sqrt{a} < \frac{b-a}{2\sqrt{a}}$$

Let $f(x) = \sqrt{x}$ and consider $f(x)$ on the interval $[a, b]$. We know $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Therefore, the Mean Value Theorem gives $c \in (a, b)$ such that
$$f(b) - f(a) = f'(c)(b-a)$$
$$\sqrt{b} - \sqrt{a} = \frac{1}{2\sqrt{c}}(b-a)$$

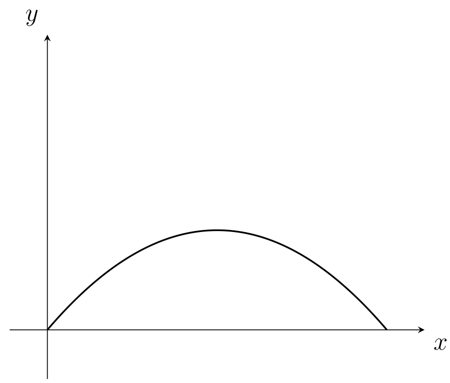
But $\frac{1}{2\sqrt{x}}$ is largest when x is as small as possible. Then on the interval $[a, b]$, this is when $x = a$. But then
$$\sqrt{b} - \sqrt{a} = \frac{1}{2\sqrt{c}}(b-a) < \frac{1}{2\sqrt{a}}(b-a) = \frac{b-a}{2\sqrt{a}}$$

(b) Is there a differentiable function on $[-1, 2]$ such that $f(-1) = -2$, $f(2) = 5$, and $f'(x) < 2$ for all $x \in [-1, 2]$? Justify your answer completely.

Assume there is such a function. We know that $f(x)$ is differentiable $[-1, 2]$ so $f(x)$ is continuous on $[-1, 2]$. Then the Mean Value Theorem, gives $c \in (-1, 2)$ such that
$$f(2) - f(-1) = f'(c)\big(2 - (-1)\big)$$
$$5 - (-2) = f'(c) \cdot 3$$
$$7 = 3f'(c)$$
$$f'(c) = \frac{7}{3} > 2$$

But $f'(x) < 2$ on $[-1, 2]$, a contradiction. So no such $f(x)$ can exist.

4. (10 points) If a projectile is fired at an initial velocity of v_0 at an angle θ—neglecting air resistance—the projectile follows a parabolic path. A simple calculation shows that at time t, the x-coordinate of the projectile is given by $x(t) = tv_0 \cos\theta$ and the y-coordinate is given by $y(t) = tv_0 \sin\theta - gt^2/2$, where g is the acceleration due to gravity.



(a) Solve for t in $x(t)$ and use this to find y in terms of x.

We know $t = \dfrac{x}{v_0 \cos\theta}$. *But then* $y = v_0 \sin\theta \cdot \dfrac{x}{v_0\cos\theta} - \dfrac{g}{2}\dfrac{x^2}{v_0 \cos^2\theta}$.

(b) The range of a projectile is the distance traveled before impacting the ground. Explain why setting $y = 0$ allows you to find the range of the projectile.

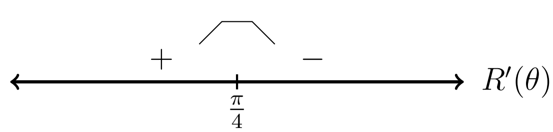
$y = 0$ *when the projective is on the ground, which only happens at the start and at the range. These are the points* $(0,0)$ *and* $(R,0)$.

(c) To find the range, set $y = 0$ and find a nonzero solution for x. Call this value R for the range and hence show $R(\theta) = \dfrac{v_0^2}{g}\sin(2\theta)$. [You will need $\sin 2\theta = 2\sin\theta\cos\theta$.]

Setting $y = 0$, *we have* $0 = v_0 \sin\theta \cdot \frac{x}{v_0\cos\theta} - \frac{g}{2}\dfrac{x^2}{v_0\cos^2\theta}$. *But then* $\frac{\sin\theta}{\cos\theta}R = \frac{g}{2}\frac{R^2}{v_0^2\cos^2\theta}$. *Then* $2\sin\theta\cos\theta v_0^2 R = gR^2$. *Using* $2\sin\theta\cos\theta = \sin(2\theta)$, *we have* $R = \frac{v_0^2 \sin 2\theta}{g}$.

(d) Find the angle θ that maximizes the range R. Be sure to justify why this is a maximum.

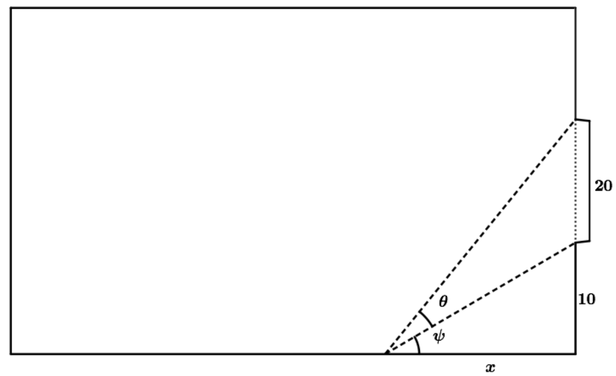
We have $\frac{dR}{d\theta} = \frac{2v_0^2\cos 2\theta}{g}$. *Setting this to 0, we find* $\frac{2v_0^2\cos 2\theta}{g} = 0$ *so that* $\cos 2\theta = 0$. *But then* $2\theta = \dfrac{\pi}{2}$ *so that* $\theta = \frac{\pi}{4}$, *i.e.* $45°$. *We check this is a maximum using the First Derivative Test:*



This confirms, neglecting wind resistance, the best angle for maximum distance is $45°$.

5. (10 points) Investigate the following problem by completing the parts below. Be sure to justify all steps.

Imagine a soccer player standing on the sideline planning to try to kick the ball into the goal. If the player stands in the bottom right corner of the field, no angle they can kick the ball at will score a goal (ignoring spin). On the other hand, if they stand in the bottom left corner of the field, they have a narrow angle window in which to kick the ball so by the time the ball "arrives" at the other side, it will land in the goal. There then must be a spot along the bottom sideline that gives the "best" possible shot in terms of the largest possible angle to kick the ball and score a goal. Here you derive this spot along the bottom of the field.



(a) Show that $\psi = \arctan\left(\frac{10}{x}\right)$.

$\tan\psi = \frac{10}{x}$ *so that* $\psi = \arctan(10/x)$.

(b) Show that $\theta + \psi = \arctan\left(\frac{30}{x}\right)$.

$\tan(\theta+\psi) = \frac{10+20}{x}$ *so that* $\theta + \psi = \arctan(30/x)$.

(c) Use the previous two parts to find the distance from the corner x that maximizes the shot angle θ. [You *do not* need to use a derivative test to prove it is a maximum.]

We know $\theta + \psi = \arctan(30/x)$ *and* $\psi = \arctan(10/x)$. *Subtracting these gives* $\theta = \arctan(30/x) - \arctan(10/x)$. *But then*

$$\theta' = \frac{1}{1+(30/x)^2} \cdot \frac{-30}{x^2} - \frac{1}{1+(10/x)^2} \cdot \frac{-10}{x^2} = \frac{-30}{x^2+900} + \frac{10}{x^2+100}$$

Setting $\theta' = 0$, *we find* $\dfrac{10}{x^2+100} = \dfrac{30}{x^2+900}$. *This implies* $3x^2 + 300 = x^2 + 900$ *so that* $x^2 = 300$ *which means* $x = \pm\sqrt{300} = \pm 10\sqrt{3}$. *Clearly,* $x > 0$ *so that* $x = 10\sqrt{3} \approx 17.32$.

(d) For this distance x, what is the maximum possible angle shot, θ?

$$\theta = \arctan\left(\frac{30}{10\sqrt{3}}\right) - \arctan\left(\frac{10}{10\sqrt{3}}\right) = \arctan\sqrt{3} - \arctan(1/\sqrt{3}) = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

Then $\theta = 30°$.