1 Definitions

Summation: The summation of a sequence of numbers $a_1, a_2, a_3, \cdots, a_n$ is $\sum_{i=1}^{n} a_i \stackrel{\text{def}}{=} a_1 + a_2 + a_3 + \cdots + a_n$. The symbol $\sum$ is called the summation, the lower number (in this case 1) is the lower index (of summation) or lower limit and the upper number (in this case n) is the upper index (of summation) or the upper limit.

Partition: A partition of an interval $[a, b]$ is a finite, ordered set of points $(x_1, x_2, \cdots, x_n)$ of points in $[a, b]$ such that $a = x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$. This partition is denoted $\mathscr{P} = \{x_i\}_{i=1}^{n}$.

Norm of a Partition: The norm (or mesh) of a partition is $\max\{x_2 - x_1, x_3 - x_2, \cdots, x_n - x_{n-1}\}$. The norm of a partition is denoted $\|\mathscr{P}\|$. If each x_i, x_{i+1} have the same difference, the partition is called regular and $\|\mathscr{P}\| = \frac{x_n - x_1}{n}$.

Tagged Partition: If a point t_i has been selected from each subinterval $I_i = [x_i, x_{i+1}]$ for $i = 1, 2, \cdots, n-1$ for a partition of $[a, b]$, then the set of ordered pairs $\dot{\mathscr{P}} = \{([x_i, x_{i+1}], t_i)\}_{i=1}^{n-1}$ is called a tagged partition of $[a, b]$.

Riemann Sum: Given a tagged partition of $[a, b]$ and a function $f(x)$, a Riemann Sum of $f : [a, b] \to \mathbb{R}$ corresponding to the tagged partition is the number

$$S(f; \dot{\mathscr{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

Riemann Integrable: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable on $[a, b]$ if there is $L \in \mathbb{R}$ such that for all $\epsilon > 0$, there is a $\delta > 0$ such that if $\dot{\mathscr{P}}$ is any tagged partition of $[a, b]$ with $\|\dot{\mathscr{P}}\| < \delta$, then $|S(f; \dot{\mathscr{P}}) - L| < \epsilon$. The set of Riemann integrable functions on $[a, b]$ is denoted $\mathscr{R}[a, b]$. If $f \in \mathscr{R}[a, b]$, then L is uniquely determine and in place of L one writes

$$L = \int_a^b f(x)\, dx$$

Riemann Integrable ("Casual Version"): If $f(x)$ is defined on $[a, b]$ and the limit

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i$$

exists, where $\|\Delta\|$ is the norm of the partition, then f is said to be integrable on $[a, b]$ and the limit is denoted

$$\int_a^b f(x)\, dx \stackrel{\text{def}}{=} \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i$$

2 Examples

2.1 Summation

Example 2.1.

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

Example 2.2.

$$\sum_{i=1}^{100} \frac{1}{i^2+1} - \frac{1}{5i+1} = \left(\frac{1}{1^2+1} - \frac{1}{5(1)+1}\right) + \left(\frac{1}{2^2+1} - \frac{1}{5(2)+1}\right) + \cdots + \left(\frac{1}{100^2+1} - \frac{1}{5(100)+1}\right)$$
$$\approx 0.0865862$$

Example 2.3.

$$\frac{1}{n}\sum_{i=1}^{n} b_i = \frac{b_1 + b_2 + b_3 + \cdots + b_{n-1} + b_n}{n}$$

2.2 Partition

Example 2.4. *If* $I = [0,10]$*, then* $\mathscr{P} = \{0,1,2,3,4,5,6,7,8,9,10\}$ *is a partition of* $[0,10]$*. Another partition of* I *is* $\mathscr{P} = \{0,0.5,1,1.5,2,2.5,3,3.5,4,4.5,5,5.5,6,6.5,7,7.5,8,8.5,9,9.5,10\}$*. Yet another partition is* $\mathscr{P} = \{0,1.5,5,6,7,10\}$*.*

Example 2.5. *If* $I = [-1,2]$*, then* $\mathscr{P} = \{-1,-0.5,0,1.5,2\}$ *is a partition of* $[-1,2]$ *with subintervals* $[-1,-0.5],[-0.5,0],[0,1.5],[1.5,2]$*.*

2.3 Norm of a Partition

Example 2.6. *If* $I = [0,10]$*, then the norm of the partition* $\mathscr{P} = \{0,1,2,3,4,5,6,7,8,9,10\}$ *is 1, i.e.* $\|\mathscr{P}\| = 1$*. However, the partition* $\mathscr{P}' = \{0,1.5,5,6,7,10\}$ *has norm* $\|\mathscr{P}'\| = 5 - 1.5 = 3.5$*.*

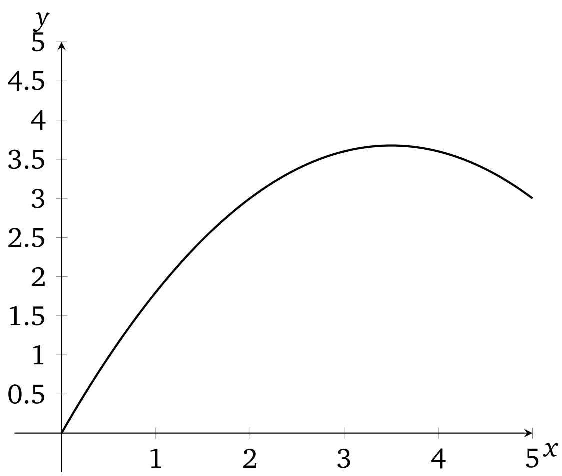
2.4 Tagged Partition

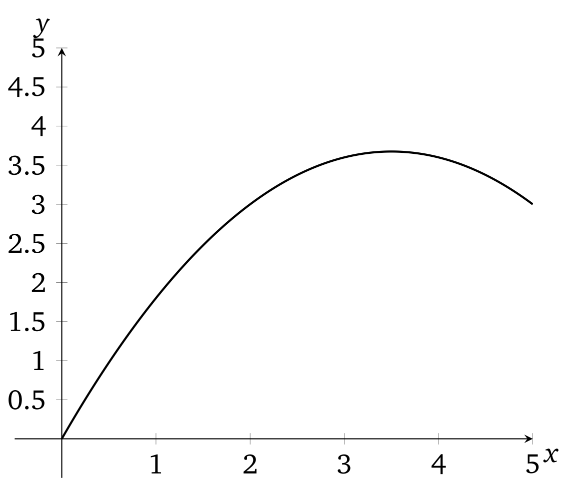
Example 2.7. *If* $I = [0,3]$*, then* $\mathscr{P} = \{0,1,1.5,2,3\}$ *is a partition of* I *with subintervals* $[0,1],[1,1.5],[1.5,2],[2,3]$*. Then* $\{([0,1],0.2122),([1,1.5],1.3499),([1.5,2],2),([2,3],2)\}$ *is a tagged partition of* I*.*

3 Riemann Sums

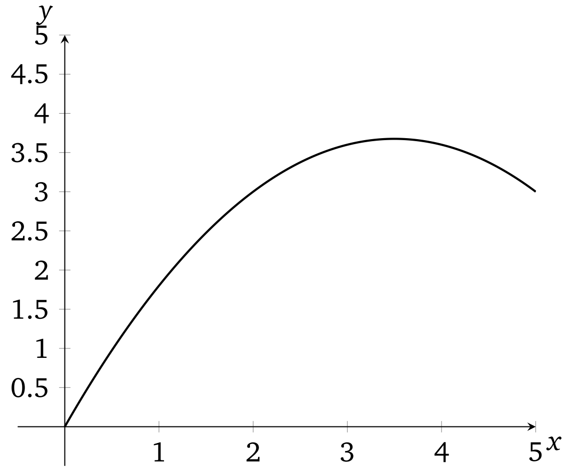
Consider $[a, b]$ with regular partition $\{a = x_1, x_2, x_3, \cdots, x_n = b\}$, i.e. $\|\Delta\| \stackrel{\text{def}}{=} \Delta x = \dfrac{b-a}{n}$.

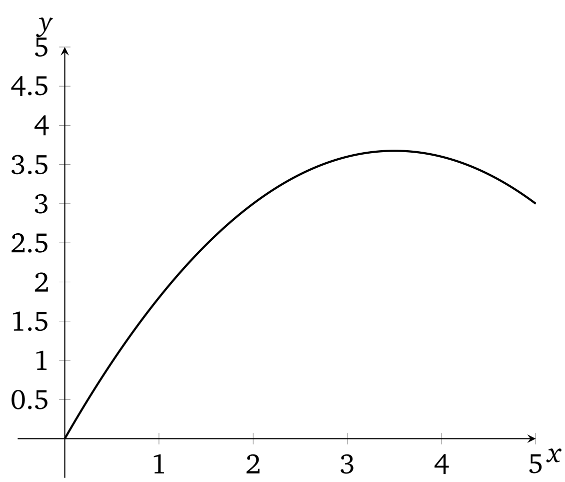
The Left-Hand Sum (LHS): $\displaystyle\sum_{i=1}^{n-1} f(x_i)\,\Delta x = \frac{b-a}{n}\left(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right)$



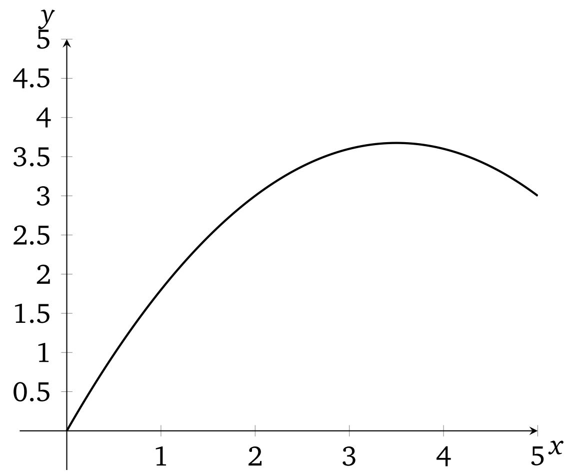


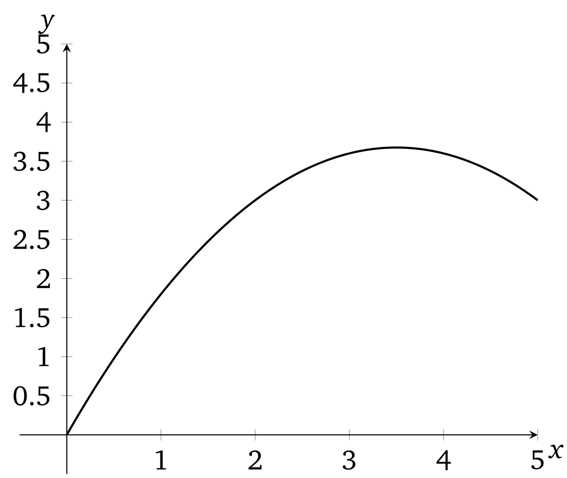
The Right-Hand Sum (RHS): $\sum_{i=2}^{n} f(x_i)\,\Delta x = \frac{b-a}{n}\left(f(x_2)+f(x_3)+\cdots+f(x_n)\right)$



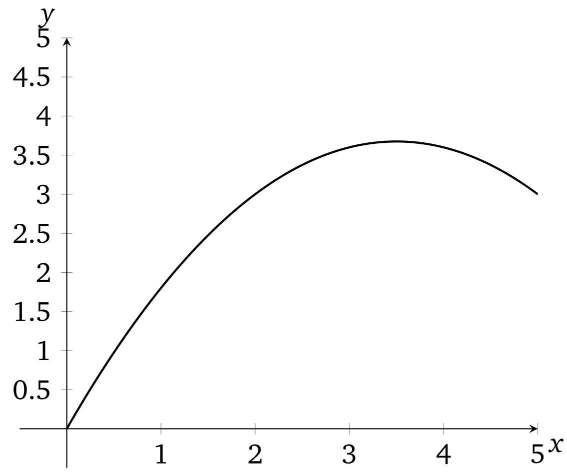


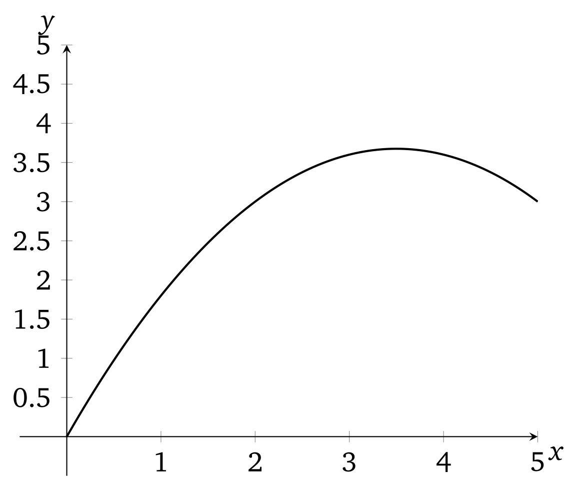
The Midpoint Sum (MS): $\sum_{i=1}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \Delta x = \frac{b-a}{n}\left(f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right)$





The Trapezoidal Rule (TS): $\dfrac{b-a}{2n}\left(f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+\cdots+2f(x_{n-1})+f(x_n)\right)=\dfrac{\text{LHS}\ +\ \text{RHS}}{2}$





Simpson's Rule (n must be even): $\frac{b-a}{3n}\big(f(x_1)+4f(x_2)+2f(x_3)+4f(x_4)+\cdots+f(x_n)\big)=\frac{2\ \text{MS}+\ \text{TS}}{3}$