Name: *Caleb McWhorter — Solutions* **Quiz 10**
MAT 295 **Fall 2016**

Problem 1: Verify that the function $f(x) = \dfrac{x-1}{x+2}$ satisfies the condition for the Mean Value Theorem on $[0,2]$ and find any points $c \in [0,2]$ satisfying the condition guaranteed by the theorem.

Because $f(x)$ is a rational function, $f(x)$ is continuous and differentiable everywhere it is defined, i.e. on every interval not containing $x = -2$. Therefore, $f(x)$ is continuous on $[0,2]$ and differentiable on $(0,2)$. Therefore, the Mean Value Theorem applies: there exists $c \in (0,2)$ with $f(2)-f(0) = f'(c)(2-0)$. Now $f(2) = \frac{1}{4}$ and $f(0) = -\frac{1}{2}$. We know that $f'(x) = \dfrac{(x+2)-1(x-1)}{(x+2)^2} = \dfrac{3}{(x+2)^2}$. Then

$$\begin{aligned}
\frac{1}{4} - \frac{-1}{2} &= 2 \cdot \frac{3}{(c+2)^2} \\
\frac{3}{4} &= \frac{6}{(c+2)^2} \\
(c+2)^2 &= 8 \\
c+2 &= \pm 2\sqrt{2} \\
c &= -2 \pm 2\sqrt{2} = 2(-1 \pm \sqrt{2})
\end{aligned}$$

Now $2(-1-\sqrt{2}) \notin (0,2)$. Therefore, $c = 2(\sqrt{2}-1)$.

Problem 2: Find $\displaystyle\lim_{n\to\infty} \left(\cos \frac{1}{n} \right)^{2n^2}$.

Let $y = \left(\cos \frac{1}{n} \right)^{2n^2}$. Then $\ln y = \ln \left(\cos \frac{1}{n} \right)^{2n^2} = 2n^2 \ln \left(\cos \frac{1}{n} \right)$. Now

$$\begin{aligned}
\lim_{n\to\infty} 2n^2 \ln \left(\cos \frac{1}{n} \right) &= \lim_{n\to\infty} \frac{2\ln(\cos 1/n)}{1/n^2} \\
&\overset{L.H.}{=} \lim_{n\to\infty} \frac{2\dfrac{-\sin(1/n)}{\cos(1/n)} \cdot \dfrac{-1}{n^2}}{-2/n^3} \\
&= \lim_{n\to\infty} \frac{-\tan(1/n)}{1/n} \\
&\overset{L.H.}{=} \lim_{n\to\infty} \frac{-\sec^2(1/n) \cdot -1/n^2}{-1/n^2} \\
&= \lim_{n\to\infty} -\sec^2(1/n) = -1
\end{aligned}$$

But then $\ln y = -1$ so that $y = e^{-1} = \dfrac{1}{e}$. Therefore, $\displaystyle\lim_{n\to\infty} \left(\cos \frac{1}{n} \right)^{2n^2} = \frac{1}{e}$.

Problem 3: Assuming a classical model of the atom, Niels Bohr was able to show that the energy of a hydrogen atom with separation r between the proton and the election is given by

$$E(r) = \frac{\hbar^2}{2m_e r^2} - \frac{e^2}{4\pi\epsilon_0 r}$$

where $\hbar$ is the reduced Planck's constant (Dirac constant), m_e is the mass of the election, e is the charge of an electron, and ϵ_0 is permittivity of free space. The Bohr radius for the hydrogen atom, denoted r_{Bohr}, is the radius at which $E(r)$ is minimal and it is approximately the expected distance between the proton and the electron in the ground state. Show that

$$r_{\text{Bohr}} = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$$

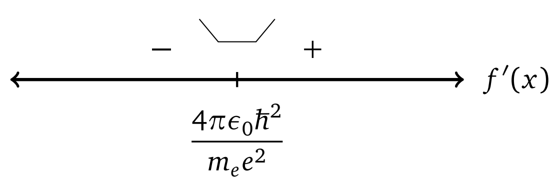
$$E(r) = \frac{\hbar^2}{2m_e r^2} - \frac{e^2}{4\pi\epsilon_0 r}$$

$$E'(r) = \frac{-2\hbar^2}{2m_e r^3} + \frac{e^2}{4\pi\epsilon_0 r^2} = \frac{e^2}{4\pi\epsilon_0 r^2} - \frac{\hbar^2}{m_e r^3}$$

If r is minimum value, then $E'(r) = 0$. But then

$$\begin{aligned} \frac{e^2}{4\pi\epsilon_0 r^2} - \frac{\hbar^2}{m_e r^3} &= 0 \\ \frac{e^2}{4\pi\epsilon_0 r^2} &= \frac{\hbar^2}{m_e r^3} \\ \frac{e^2 r}{4\pi\epsilon_0} &= \frac{\hbar^2}{m_e} \\ r &= \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} \end{aligned}$$

We check that this is a minimum using the First Derivative Test:



Therefore, $r_{Bohr} = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$.