Name: *Caleb McWhorter — Solutions* **Quiz 11**
MAT 295 **Fall 2016**

Problem 1: Find the tangent line to the curve given by the set of (x, y) satisfying $x^2 + \sin y = xy^2 + 1$ at the point $(1, 0)$.

$$\begin{aligned} x^2 + \sin y &= xy^2 + 1 \\ \frac{d}{dx}\left(x^2 + \sin y\right) &= \frac{d}{dx}\left(xy^2 + 1\right) \\ 2x + \cos y \cdot y' &= y^2 + 2xy \cdot y' \\ \cos y \cdot y' - 2xy \cdot y' &= y^2 - 2x \\ y'(\cos y - 2xy) &= y^2 - 2x \\ \frac{dy}{dx} &= \frac{y^2 - 2x}{\cos y - 2xy} \end{aligned}$$

But then $\left.\dfrac{y^2 - 2x}{\cos y - 2xy}\right|_{(1,0)} = \dfrac{0 - 2(1)}{1 - 2(1)0} = \dfrac{-2}{1} = -2$. *Therefore,* $\ell(x) = 0 + (-2)(x - 1) = -2(x - 1) = 2(1 - x)$, *i.e.* $\ell(x) = 2(1 - x)$.

Problem 2: Evaluate the following limit in two different ways: using l'Hôpital's Rule and using algebra prestidigitation

$$\lim_{x \to 0} (\csc x - \cot x)$$

Using l'Hôpital's Rule, we have

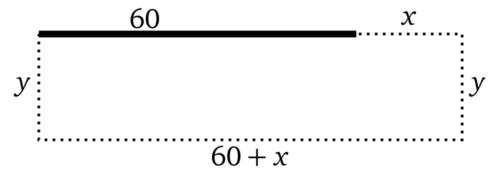
$$\lim_{x \to 0} (\csc x - \cot x) = \lim_{x \to 0} \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x} \overset{L.H.}{=} \lim_{x \to 0} \frac{\sin x}{\cos x} = \lim_{x \to 0} \tan x = 0$$

Now using an Algebra 'trick', we have

$$\lim_{x \to 0} (\csc x - \cot x) = \lim_{x \to 0} \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x} \cdot \frac{x}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x} \cdot \frac{x}{\sin x} = 1 \cdot 0 = 0$$

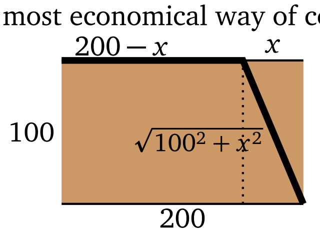
where we have used the fact that $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ *and* $\lim_{x \to 0} \dfrac{x}{\sin x} = 1$.

Problem 3: Suppose one day on the farm while leaning against a 60 ft long fence, you notice extra mesh fencing left over by the tractor shed—about 100 ft worth if you recall. Thinking you should put the fencing to good use. Given your compulsion for restricting the locomotive abilities of the chickens on your farm, you decide to attach the mesh to the ends of the fence and create an enclosure for the chickens in the manner pictured below. What is the maximum area of the enclosure you can create?



The area is given by $A = y(60+x)$. *But we know that* $100 = x + y + 60 + x + y = 2x + 2y + 60$ *so that* $y = 20 - x$. *But then we have* $A(x) = y(x+60) = (20-x)(60+x)$. *As* $y \geq 0$, *we know that* $x \in [0, 20]$. *Now* $A'(x) = -40 - 2x$. *If* $A'(x) = 0$, *then* $-40 - 2x = 0$ *so that* $x = -20$. *But* $x = -20 \notin [0, 20]$. *Therefore, the maximum area but be one of the interval endpoints. We have* $A(0) = 20(60) = 1200$ *and* $A(20) = 0(80) = 0$. *As* $A'(x) < 0$ *on* $[0, 20]$, *it must be that* $A(0) = 1200$ *sq.ft is the maximum area.*

Problem 4: A natural gas pipe is to be laid connecting the underground gas reserve, located 100 meters underground, to the processing facility 200 meters away horizontally on the surface. Ordinary surface pipe costs roughly \$2,500 per meter in total construction cost while laying the pipe underground—requiring tedious construction and more reinforcement—requires approximately \$5,000 per meter. Find the most economical way of constructing the pipeline.



Let $C(x)$ denote the cost. For simplicity of notation, let $m = 2500$ and $M = 5000$. Now we have $C(x) = m(200 - x) + M\sqrt{100^2 + x^2}$. Then $C'(x) = -m + \dfrac{Mx}{\sqrt{100^2+x^2}} = 0$. Setting $C'(x) = 0$, we have...

$$\begin{aligned}
m &= \frac{Mx}{\sqrt{100^2+x^2}} \\
Mx &= m\sqrt{100^2+x^2} \\
M^2x^2 &= m^2(100^2+x^2) \\
M^2x^2 - m^2x^2 &= 100^2m^2 \\
x^2 &= \frac{100^2m^2}{M^2-m^2} \\
x &= \pm\frac{100m}{\sqrt{M^2-m^2}}
\end{aligned}$$

Now clearly $x > 0$ so that $x = \dfrac{100m}{\sqrt{M^2-m^2}} = \dfrac{100 \cdot 2500}{\sqrt{5000^2 - 2500^2}} = \dfrac{100}{\sqrt{3}}$. Then $C(100/\sqrt{3}) = \$933,013.70$.