Name: *Caleb McWhorter — Solutions* **Quiz 3**
MAT 295 **Fall 2016**

Problem 1: Use the Squeeze Theorem to show that $\lim_{x\to 0} x^3\, 2^{\cos(1/x)} = 0$.

Solution. *We know that* $-1 \leq \cos(1/x) \leq 1$. *Because* 2^x *is monotonically increasing, this implies* $2^{-1} \leq 2^{\cos(1/x)} \leq 2^1$. *Therefore,*

$$\begin{aligned} x^3 2^{-1} \leq &\quad x^3 2^{\cos(1/x)} \quad \leq x^3 \cdot 2 \\ \frac{x^3}{2} \leq &\quad x^3 2^{\cos(1/x)} \quad \leq 2x^3 \end{aligned}$$

But $\lim_{x\to 0} \frac{x^3}{2} = 0 = \lim_{x\to 0} 2x^3$. *Therefore by Squeeze Theorem,* $\lim_{x\to 0} x^3 2^{\cos(1/x)} = 0$.

Problem 2: Use the Intermediate Value Theorem to show that $f(x) = \pi x^2 e^{-x} - 1$ has a root on the interval $[0,1]$.

Solution. *The function* $f(x)$ *is continuous. Observe that*

$$\begin{aligned} f(0) &= \pi \cdot 0 e^0 - 1 = -1 < 0 \\ f(1) &= \pi \cdot 1 \cdot e^{-1} - 1 = \frac{\pi}{e} - 1 > 0 \end{aligned}$$

where the last inequality follows from the fact that $\pi > e$ *(*$\pi \approx 3$ *and* $e \approx 2$*). Because* $-1 < 0 < \frac{\pi}{e} - 1$, *it follows from the Intermediate Value Theorem that there is* $c \in [0,1]$ *so that* $f(c) = 0$. *But then* c *is a root of* $f(x)$ *on* $[0,1]$.

Problem 3: Use the definition of the derivative to find the derivative of $f(x) = 2x^2 + 3x - 5$.

$$f'(x) \stackrel{def}{=} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left[2(x+h)^2 + 3(x+h) - 5\right] - (2x^2 + 3x - 5)}{h}$$

$$= \lim_{h\to 0} \frac{2(x^2 + 2hx + h^2) + 3x + 3h - 2x^2 - 3x + 5}{h}$$

$$= \lim_{h\to 0} \frac{2x^2 + 4hx + 2h^2 + 3x + 3h - 2x^2 - 3x + 5}{h}$$

$$= \lim_{h\to 0} \frac{4hx + 2h^2 + 3h}{h}$$

$$= \lim_{h\to 0} \frac{h(4x + 2h + 3)}{h}$$

$$= \lim_{h\to 0} (4x + 2h + 3)$$

$$= 4x + 0 + 3$$

$$= 4x + 3$$