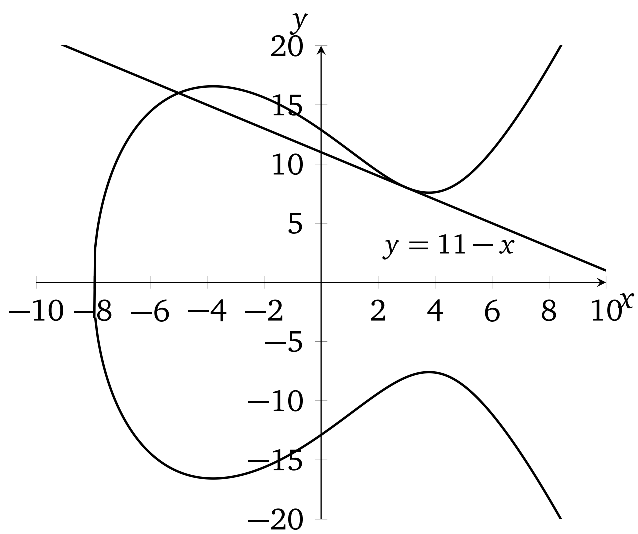
Name: *Caleb McWhorter — Solutions* **Quiz 6**
MAT 295 **Fall 2016**

Problem: Consider the elliptic curve[1] $y^2 = x^3 - 43x + 166$, which is plotted below.

(a) Use implicit differentiation to find the equation of the tangent line to the rational point $(3, 8)$ on the elliptic curve.

(b) Sketch the tangent line from (a) on the given plot.

(c) Use the plot from (b) to determine if the tangent line would give a good approximation for the y-coordinate of the curve at $x = 2$.

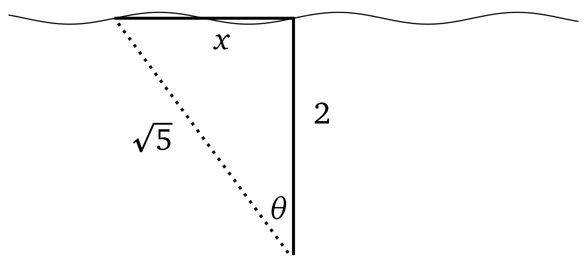


$$
\begin{aligned}
y^2 &= x^3 - 43x + 166 \\
\frac{d}{dx}\, y^2 &= \frac{d}{dx}(x^3 - 43x + 166) \\
2y \cdot y' &= 3x^2 - 43 \\
\frac{dy}{dx} &= \frac{3x^2 - 43}{2y}
\end{aligned}
$$

But then $\left.\dfrac{3x^2 - 43}{2y}\right|_{(3,8)} = \dfrac{3(3^2) - 43}{2(8)} = -1$. *But then the tangent line is* $y = 8 + (-1)(x - 3)$ *so that* $y = 11 - x$. *Notice from the plot, we do not expect the tangent line to approximate the value at* $x = 2$ *very well. It is even worse for* $x = 1, 0$, *etc. Note that the error of the approximation at* $x = 2$ *is 4% but at* $x = 1$ *it is 10%.*

[1] Elliptic curves are curves (of degree 2) which are of the form $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$ and form an essential part of modern day cryptography and Mathematics generally. Faltings was awarded the Fields Medal—the highest possible honor in Mathematics—for proving there are only finitely many rational points on elliptic curves.

Problem 1: A "**Caution: Manatee Area**" warning light is located on a small buoy 2 km away from a beach. The light located on the buoy rotates in place on the buoy at a rate of 2 times per minute. How fast is the light racing across the shoreline when the light is facing toward a point on the shore 1 km away from spot on the shore closest to the warning light?



Using the Pythagorean Theorem, we have $a^2 + b^2 = c^2$*. But then* $c^2 = 1^2 + 2^2 = 5$ *so that* $c = \sqrt{5}$*. The light rotates 2 times per minute so that it rotates at a rate of...*

$$\underbrace{2}_{\#\ rotations} \cdot \underbrace{\frac{2\pi}{1\ min}}_{rad/min} = 4\pi\ rad/min$$

Now $\tan\theta = \dfrac{x}{2}$ *so that* $x = 2\tan\theta$*. But then*

$$\underbrace{\frac{dx}{dt}}_{horz\ velocity} = 2\sec^2\theta \cdot \underbrace{\frac{d\theta}{dt}}_{angular\ velocity}$$

We know that $\dfrac{d\theta}{dt} = 4\pi$*. Using right triangle trig, we know also that* $\sec\theta = \dfrac{\sqrt{5}}{2}$*. But then*

$$\begin{aligned} \frac{dx}{dt} &= 2\sec^2\theta \cdot \frac{d\theta}{dt} \\ x' &= 2\left(\frac{\sqrt{5}}{2}\right)^2 \cdot 4\pi \\ x' &= 2 \cdot \frac{5}{2^2} \cdot 4\pi \\ x' &= 10\pi\ km/min \end{aligned}$$

We can also check dimensions:

$$x' = \underbrace{2}_{m} \cdot \underbrace{\sec^2\theta}_{unitless} \cdot \underbrace{\frac{d\theta}{dt}}_{1/min}$$