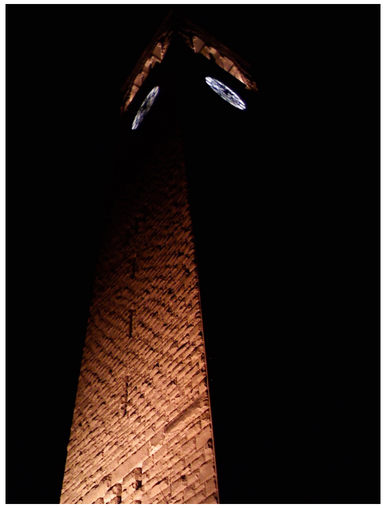
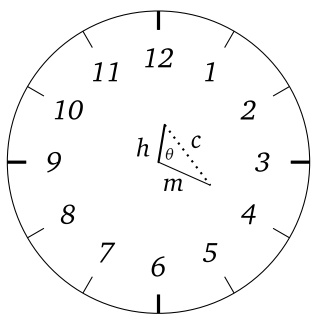
Name: *Caleb McWhorter — Solutions* **Quiz 7**

MAT 295 **Fall 2016**

Problem 1: The McGraw Clock Tower at Cornell University has minute hand and hour hands roughly 5 ft and 3 ft in length, respectively. If a student walked by Olin Library at 12:19am and looked up at the clock, what would they measure the rate of change of the distance between the tips of the hands of the clock to be?





We need the angle between the hands at any given time. We use $d = vt$, i.e. total = rate × time, where d is the angle, v is the angular velocity, and t is a time. Let m denote the minute hand length, h denote the hour hand length, c denote the distance between the hand tips, M denote the minute time, H denote the hour time, and θ denote the angle between the hands.

Using the Law of Cosines, $c^2 = m^2 + h^2 - 2mh\cos\theta = 5^2 + 3^2 - 2(5)3\cos\theta = 34 - 30\cos\theta$. Differentiating, we have $2c\dfrac{dc}{dt} = 30\sin\theta\dfrac{d\theta}{dt}$ so that $\dfrac{dc}{dt} = \dfrac{15\sin\theta}{c}\,\dfrac{d\theta}{dt}$.

Now we only need to find the angle and rate of change of the angle. Note that

$$v_M = \frac{d}{t} = \frac{2\pi}{60 \text{ min}} = \frac{\pi}{30} \text{ rad/min}$$

$$v_H = \frac{d}{t} = \frac{2\pi}{12 \text{ hr}} = \frac{\pi \text{ rad}}{6 \text{ hr}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} = \frac{\pi}{360} \text{ rad/min}$$

The angle 'formed' by the minute hand is then $\dfrac{\pi}{30}M$. The angle 'formed' by the hour hand is $\dfrac{\pi}{6}H + \dfrac{\pi}{360}M$.

Then the angle between them is

$$\theta = \frac{\pi}{30}M - \left(\frac{\pi}{6}H + \frac{\pi}{360}M\right) = \frac{11\pi}{360}M - \frac{\pi}{6}H$$

It is then clear that $\frac{d\theta}{dt} = v_M - v_H = \frac{\pi}{30} - \frac{\pi}{360} = \frac{11\pi}{360}$ *rad/min. It is 12:19 am, so* $H = 0$ *and* $M = 19$. *Then*

$$\theta = \frac{11\pi}{360}\cdot 19 - \frac{\pi}{6}\cdot 0 = \frac{209\pi}{360}$$

$$\frac{d\theta}{dt} = \frac{11\pi}{360}$$

$$c^2 = 34 - 30\cos\left(\frac{209\pi}{360}\right) = 41.5114 \Rightarrow c \approx 6.4429\,ft$$

$$\frac{dc}{dt} = \frac{15\sin\left(\frac{209\pi}{360}\right)}{6.4429}\cdot\frac{11\pi}{360} \approx 0.216367\,ft/min$$

We can also check dimensions:

$$\underbrace{\frac{dc}{dt}}_{ft/min} = \frac{ft^2}{ft}\cdot\frac{1}{min} = ft/min$$

Note that this can be done in degrees but one must be careful! Note that

$$\frac{d\theta}{dt} \overset{def}{=} \frac{d\theta_{rad}}{dt} = \frac{d}{dt}\theta_{rad} = \frac{d}{dt}\left(\theta_{deg}\frac{\pi}{180}\right) = \frac{\pi}{180}\frac{d\theta_{deg}}{dt}$$

Furthermore, derivative then do not work the traditional way: $\frac{d}{d\theta}\sin\theta = \cos\theta$ *if* θ *is measured in radians, but otherwise there is a chain rule:*

$$\frac{d}{d\theta}\sin\theta_{rad} = \frac{d}{d\theta}\sin\left(\theta_{deg}\cdot\frac{\pi}{180}\right) = \frac{\pi}{180}\cos\left(\theta_{deg}\cdot\frac{\pi}{180}\right)$$

One would then find $\theta = 5.5M - 30H = \frac{11M - 60H}{2}$, $v_M = 6°/min$, $v_H = 0.5°/min$, $\theta = 104.5°$ *and* $\frac{dc}{dt} = \frac{dc}{d\theta}\cdot\frac{d\theta_{rad}}{dt}$.

Problem 2: In Newtonian Physics for particles, $F = ma$ where F is the force, m is the mass, and a is the acceleration of the particle. But $F = ma$ does not always hold. Generally, $F = \frac{dp}{dt}$, where $p = mv$ is the momentum of the particle at time t.

(a) Suppose that a particles mass is essentially constant and that its velocity is not "too large." Use $F = \frac{dp}{dt}$ and $p = mv$ to show that $F = ma$.

$$F = \frac{d}{dt}p = \frac{d}{dt}mv = \underbrace{\frac{dm}{dt}}_{\approx 0} v + m\underbrace{\frac{dv}{dt}}_{a} = 0 + ma = ma$$

For particles moving at a high velocity, $F = ma$ is no longer valid. In Special Relativity, the momentum of a particle is given by

$$p = \gamma mv = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m is the (rest) mass of the particle and c is the speed of light in a vacuum.

(b) What happens to the momentum of a particle as v approaches c? Can an object with mass travel at the speed of light?[1]

$$\lim_{v \to c} p = \lim_{v \to c} \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} = \infty$$

It is impossible for a particle to have infinite momentum (except for light); therefore, no object with mass can travel at the speed of light.

(c) Given $F = \dfrac{dp}{dt}$ still holds in Special Relativity, show that the relativistic force for a particle with rest mass m moving at velocity v is given by

$$F = \frac{ma}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}$$

Use this to show why it is impossible to accelerate an object with nonzero mass to the speed of light.

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[1]Note: sometimes the particle is said to have mass $M = \gamma m$, called the relativistic mass. Then the rest mass is when $v = 0$ and $p = Mv$. However, this is convention; *Mass is an intrinsic, invariant quantity that does not depend on velocity.*]

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\begin{aligned}
F &= \frac{d}{dt}\,p \\
&= \frac{d}{dt}(\gamma m v) \\
&= \frac{d\gamma}{dt} m v + \gamma \underbrace{\frac{dm}{dt}}_{\approx 0} v + \gamma m \underbrace{\frac{dv}{dt}}_{a} \\
&= \frac{-1}{2} \cdot \frac{mv \cdot \frac{-2}{c^2} \cdot v v'}{(1 - v^2/c^2)^{3/2}} + 0 + \gamma m a \\
&= \frac{mav^2}{c^2(1 - v^2/c^2)^{3/2}} + \frac{ma}{\sqrt{1 - v^2/c^2}} \cdot \frac{c^2}{c^2} \cdot \frac{1 - v^2/c^2}{1 - v^2/c^2} \\
&= \frac{mav^2 + mac^2 - mav^2}{c^2(1 - v^2/c^2)^{3/2}} \\
&= \frac{mac}{c^2(1 - v^2/c^2)^{3/2}} \\
&= \frac{ma}{(1 - v^2/c^2)^{3/2}}
\end{aligned}
$$

It is clear that $\lim_{v \to c} F = \lim_{v \to c} \frac{ma}{(1 - v^2/c^2)^{3/2}} = \infty$. *Of course, one cannot apply infinite force to an object so that one cannot accelerate objects with mass to the speed of light.*