Math 397 Spring 2016 Exam I Solutions

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Problem	Points	Score
1	8	
2	8	
3	10	
4	8	
5	10	
6	10	
Total	60	

Do all your work on this exam. Correct answers should be supported by your calculations and reasoning where appropriate.

1. (a) Find all unit vectors parallel to $\bar{v} = \langle 1, -3, \sqrt{2}\rangle$.

$|\vec{v}| = \sqrt{1^2 + 3^2 + \sqrt{2}^2} = \sqrt{1+9+2} = \sqrt{12}$

Then the only two such vectors are

$\left\langle \frac{1}{\sqrt{12}}, \frac{-3}{\sqrt{12}}, \sqrt{\frac{1}{6}} \right\rangle$ and $\left\langle \frac{-1}{\sqrt{12}}, \frac{3}{\sqrt{12}}, -\sqrt{\frac{1}{6}} \right\rangle$

$\frac{\sqrt{2}}{\sqrt{12}}$

(b) Which, if any, of the following vectors is orthogonal to $\bar{v} = \langle 2, -3, \sqrt{2}\rangle$? (Show your computations.)

$\bar{a} = \langle 3, 2, 5\sqrt{2}\rangle$, $\bar{b} = \langle -6, 2, 6\sqrt{2}\rangle$

$\vec{v} \cdot \vec{a} = 2(3) + -3(2) + \sqrt{2} \cdot 5\sqrt{2} = 6 - 6 + 10 = 10 \neq 0$

$\vec{v} \cdot \vec{b} = 2(-6) + -3(2) + \sqrt{2} \cdot 6\sqrt{2} = -12 - 6 + 12 = -6 \neq 0$

There $\vec{v}$ is orthogonal to neither $\vec{a}$ nor $\vec{b}$.

(c) Assume $\bar{u} \bullet \bar{v} = 8$ and $\bar{u} \bullet \bar{w} = -5$ find

(i) $\bar{u} \bullet (3\bar{v} + 2\bar{w}) = 3(\vec{u} \cdot \vec{v}) + 2(\vec{u} \cdot \vec{w}) = 3(8) + 2(-5) = 24 - 10 = 14$

(ii) For what value of k is $\bar{v} + k\bar{w}$ orthogonal to $\bar{u}$?

$0 = \vec{u} \cdot (\vec{v} + k\vec{w}) = \vec{u} \cdot \vec{v} + k(\vec{u} \cdot \vec{w})$

$= 8 + k(-5)$

$= 8 - 5k$

So $k = 8/5$

2. (a) Find the point of intersection of the following pair of lines.

$$L_1: \begin{array}{l} x = 1+t \\ y = 2+2t \\ z = 4-t \end{array} \quad -\infty < t < \infty \quad \text{and} \quad L_2: \begin{array}{l} x = 4-s \\ y = 2+s \\ z = 2s-1 \end{array} \quad -\infty < s < \infty$$

$$\left.\begin{array}{l} 1+t = 4-s \\ 2+2t = 2+s \end{array}\right\} \longrightarrow \begin{array}{l} 1+t=4-s \\ 2+2t=2+s \end{array} \longrightarrow \begin{array}{r} t+s=3 \\ +\quad 2t-s=0 \\ \hline 3t=3 \end{array}$$

$$4-t = 2s-1$$

$t=1 \rightarrow$ so...

$$\begin{aligned} t+s&=3 \\ 1+s&=3 \\ s&=2 \end{aligned}$$

$$\begin{aligned} 4-t &\overset{?}{=} 2s-1 \\ 4-1 &\overset{?}{=} 2(2)-1 \\ 3 &= 3 \; \checkmark \end{aligned}$$

$$\begin{aligned} x &= 1+t = 2 \\ y &= 2+2t = 4 \\ z &= 4-t = 3 \end{aligned}$$

$(2, 4, 3)$

(b) Find the angle between the lines in part (a).

$L_1: \langle 1, 2, -1 \rangle \overset{def}{=} \vec{u}$

$L_2: \langle -1, 1, -1 \rangle \overset{def}{=} \vec{v}$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 1(-1) + 2(1) + (-1)(-1) \\ &= -1 + 2 + 1 \\ &= 2 \end{aligned}$$

$|\vec{u}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$

$|\vec{v}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= |\vec{u}|\,|\vec{v}| \cos\theta \\ 2 &= \sqrt{6}\sqrt{3} \cos\theta \\ 2 &= 3\sqrt{2} \cos\theta \\ \cos\theta &= \frac{2}{3\sqrt{2}} \\ \theta &= \cos^{-1}\left(2/3\sqrt{2}\right) \end{aligned}$$

3. (a) Find the area of the triangle in three-space with vertices $P(2,3,-1)$, $Q(4,5,2)$ and $R(6,2,1)$

$$PQ = \langle 4-2,\ 5-3,\ 2-{}^-1 \rangle = \langle 2, 2, 3 \rangle$$

$$\cancel{QR}\ RP = \langle 6-2,\ 2-3,\ 1-{}^-1 \rangle = \langle 4, {}^-1, 2 \rangle$$

$$PQ \times RP = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 3 \\ 4 & -1 & 2 \end{vmatrix}$$

$$= \hat{i}\,(4 - {}^-3) - \hat{j}\,(4 - 12) + \hat{k}\,(-2 - 8)$$

$$= 7\hat{i} + 8\hat{j} - 10\hat{k}$$

$$|PQ \times RP| = \sqrt{7^2 + 8^2 + 10^2} = \sqrt{49 + 64 + 100} = \sqrt{213}$$

$$\boxed{\frac{\sqrt{213}}{2}}$$

(b) Find the equation of the plane through the three points in part (a)

$$\langle 7, 8, -10 \rangle \cdot \langle x-2,\ y-3,\ z-{}^-1 \rangle = 0$$

$$7(x-2) + 8(y-3) - 10(z+1) = 0$$

$$7x + 8y - 10z - 14 - 24 - 10 = 0$$

$$7x + 8y - 10z = 48$$

4. (a) Find the parametric equations of the line that contains the point $P(3,2,-1)$ and is perpendicular to the plane $5x-4y-z+11=0$.

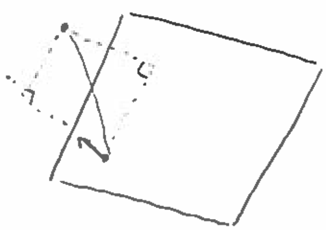
$\vec{n} = \langle 5,-4,-1\rangle$

$r(t) = \langle 5,-4,-1\rangle t + \langle 3,2,-1\rangle$

$= \langle 5t+3, -4t+2, -t-1\rangle$

$$\begin{cases} x = 5t+3 \\ y = -4t+2 \\ z = -t-1 \end{cases}$$

(b) Find the perpendicular distance between the point P and the plane in part (a).



A point on the plane is...

$\underbrace{(-2,0,1)}_{Q} \rightarrow$ as $5(-2)-4(0)-1+11=0$

$PQ = \langle 3-{-2}, 2-0, -1-1\rangle = \langle 5,2,-2\rangle$

$\text{proj}_{\vec{n}} \vec{PQ} = \dfrac{\vec{n}\cdot\vec{PQ}}{\vec{n}\cdot\vec{n}}\,\vec{n} = \dfrac{19}{42}\langle 5,-4,-1\rangle$

$\vec{n} = \langle 5,-4,-1\rangle$

$\vec{n}\cdot\vec{n} = 5^2+4^2+1^2$

$= 25+16+1$

$= 42$

$\vec{n}\cdot PQ = 5(5) + -4(2) + (-1)(-2)$

$= 25-8+2$

$= 19$

$|\text{proj}_{\vec{n}} \vec{PQ}| = \left|\dfrac{19}{42}\langle 5,-4,-1\rangle\right|$

$= \dfrac{19}{42}|\langle 5,-4,-1\rangle|$

$= \dfrac{19}{42}\sqrt{5^2+4^2+1^2}$

$= \dfrac{19}{42}\sqrt{42}$

$= \dfrac{19}{\sqrt{42}}$

5. Consider the two parallel planes $x+2y+3z=12$ containing the point $P(2,2,2)$ and $x+2y+3z=14$ containing the point $Q(5,3,1)$.

(i) Compute and <u>simplify</u> the vector projection of $\overrightarrow{PQ}$ onto the normal vector $\vec{i}+2\vec{j}+3\vec{k}=\langle 1,2,3\rangle$.

$\vec{PQ} = \langle 5-2,\ 3-2,\ 1-2\rangle = \langle 3,1,-1\rangle$

$\vec{n} = \langle 1,2,3\rangle$

$\vec{n}\cdot PQ = 1(3)+2(1)+3(-1)$
$= 3+2-3$
$= 2$

$\vec{n}\cdot\vec{n} = 1^2+2^2+3^2$
$= 1+4+9$
$= 14$

$\text{Proj}_{\vec{n}}\vec{PQ} = \dfrac{\vec{n}\cdot\vec{PQ}}{\vec{n}\cdot\vec{n}}\,\vec{n}$
$= \dfrac{2}{14}\,\vec{n}$
$= \frac{1}{7}\langle 1,2,3\rangle$
$= \langle 1/7,\ 2/7,\ 3/7\rangle$

(ii) Describe geometrically what the length of the vector you computed in part (a) gives. [A diagram may help you decide.]

The length of the vector from (a) is the distance between the planes.

6. (a) Find the center and radius of the sphere with equation $x^2-2x+y^2+6y+z^2+4z=2$.

$$x^2-2x+1+y^2+6y+9+z^2+4z+4=2+1+9+4$$

$$(x^2-2x+1)+(y^2+6y+9)+(z^2+4z+4)=16$$

$$(x-1)^2+(y+3)^2+(z+2)^2=16$$

Center: $(1,-3,-2)$

Radius: 4

(b) Give the equation of the intersection of the sphere in part (a) with the xy-plane.

In xy-plane, $z=0$

$$(x-1)^2+(y+3)^2+(0+2)^2=16$$

$$(x-1)^2+(y+3)^2+4=16$$

$$(x-1)^2+(y+3)^2=12$$

This intersection is a circle in $\mathbb{R}^3$ with...

Center: $(1,-3,0)$

Radius: $\sqrt{12}=2\sqrt{3}$