Math 397 Spring 2016 Exam III Solutions

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(Please Print.)

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

Do all your work on this exam. Correct answers should be supported by your calculations. **No calculators** may be used. **All phones must be silenced and put out of sight.**

1. (a) Find the equation of the line normal to the surface $f(x,y,z)=z^2x+zy-2x=4$ at the point $(-1,3,2)$.

$$\nabla(f(x,y,z)-4)=\nabla f(x,y,z)=\langle z^2-2,\ z,\ 2zx+y\rangle\Big|_{(-1,3,2)}$$
$$=\langle 2^2-2,\ 2,\ 2(2)(-1)+3\rangle$$
$$=\langle 2,2,-1\rangle$$

$$\begin{cases} x=2t-1 \\ y=2t+3 \\ z=-t+2 \end{cases} \quad \text{or} \quad \ell(t)=\langle 2,2,-1\rangle t+\langle -1,3,2\rangle$$

(b) In what direction does the function $f(x,y,z)=z^2x+zy-2x$ **increase** most rapidly at the point $(-1,3,2)$?

Increases most rapidly in direction $\nabla f(-1,3,2)$ which was calculated above as $\langle 2,2,-1\rangle$

2. Find all critical points of the function $f(x, y) = x^2 + 2y^2 - x^2y$ and classify them as local maxima, local minima or saddle points.

$$\begin{cases} f_x = 2x - 2xy = 0 \\ f_y = 4y - x^2 = 0 \end{cases}$$

$$\begin{cases} 2x(1-y) = 0 \\ 4y = x^2 \end{cases}$$

$$\begin{cases} x(1-y) = 0 \\ 4y = x^2 \end{cases}$$

$x(1-y) = 0$

$x = 0$ or $1 - y = 0$

$y = 1$

If $x = 0$:

$4y = x^2$

$4y = 0$

$y = 0$

$(0,0)$

If $y = 1$:

$4y = x^2$

$4 = x^2$

$x = \pm 2$

$(2,1)$ & $(-2,1)$

$f_{xx} = 2(1-y)$

$f_{yy} = 4$

$f_{xy} = f_{yx} = -2x$

$$d = \begin{vmatrix} 2(1-y) & -2x \\ -2x & 4 \end{vmatrix} = 8(1-y) - 4x^2$$

At $(0,0)$:

$d = 8(1-0) - 0 = 8 > 0$

$f_{xx} = 2(1-0) = 2 > 0$

So $(0,0)$ minimum

At $(2,1)$:

$d = 8(1-1) - 4(2^2) = -16 < 0$

So $(2,1)$ is a saddle

At $(-2,1)$:

$d = 8(1-1) - 4(-2)^2 = -16 < 0$

So $(-2,1)$ is a saddle

3. Find the maximum and minimum values for $f(x,y)=6x^2y$ subject to the condition $3x^2+2y^2=18$.

$\nabla f = \langle 12xy, 6x^2 \rangle$

$\underbrace{3x^2+2y^2-18}_{g(x,y)}=0$

$\nabla g = \langle 6x, 4y \rangle$

$\begin{cases} 12xy = \lambda 6x \\ 6x^2 = \lambda 4y \\ 3x^2+2y^2=18 \end{cases}$

$\begin{cases} 2xy = \lambda x \\ 3x^2 = 2y\lambda \\ 3x^2+2y^2=18 \end{cases}$

If $y=0 \to 6x^2=0 \to x=0$

$(0,0)$. But $3\cdot 0^2+2\cdot 0^2 \neq 18$

so $(0,0)$ fails.

If $x=0 \to \lambda y=0$ so

$y=0$ gives $(0,0)$, does not work as above, or $\lambda=0$

$3x^2+2y^2=18$

$0+2y^2=18$

$y^2=9$

$y=\pm 3$

But $x=0$ gives $f(0,-)=0$ and clearly not a solution.

If $x\neq 0, y\neq 0 \to$

$\begin{cases} 2y=\lambda \\ 3x^2=2y\lambda \\ 3x^2+2y^2=18 \end{cases}$

$3x^2=2y\lambda$

$3x^2=2y\cdot 2y$

$3x^2=4y^2$

Then....

$3x^2+2y^2=18$

$4y^2+2y^2=18$

$6y^2=18$

$y^2=3$

$y=\pm\sqrt{3}$

Then...

$3x^2+2y^2=18$

$3x^2+2(3)=18$

$3x^2=12$

$x^2=4$

$x=\pm 2$

$(2,\sqrt{3})$, $(-2,\sqrt{3})$

$(-2,\sqrt{3})$, $(-2,-\sqrt{3})$

$f(\pm 2,\sqrt{3})=6(\pm 2)^2\sqrt{3}=24\sqrt{3}$

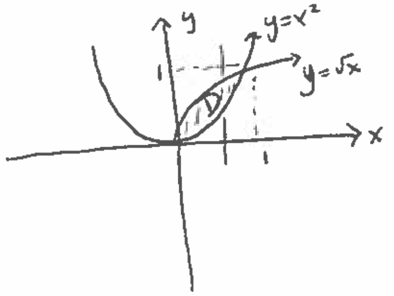
$f(\pm 2,-\sqrt{3})=6(\pm 2)^2-\sqrt{3}=-24\sqrt{3}$

So $(2,\sqrt{3})$ & $(-2,\sqrt{3})$ are max. with value $24\sqrt{3}$

&

$(2,-\sqrt{3})$ & $(-2,-\sqrt{3})$ are min with value $-24\sqrt{3}$

4. (a) Evaluate $\iint_D 3xy\, dA$ where D is the finite region bounded between the curves $y = x^2$ and $y = \sqrt{x}$.

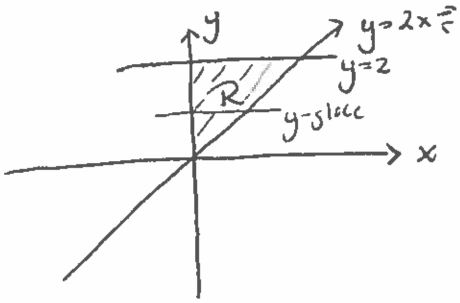


$$\iint_D 3xy\,dA = \int_0^1 \int_{x^2}^{\sqrt{x}} 3xy\,dy\,dx$$
$$= \int_0^1 3x \int_{x^2}^{\sqrt{x}} y\,dy\,dx$$
$$= \int_0^1 3x \cdot \frac{y^2}{2}\Big|_{x^2}^{\sqrt{x}}\,dx$$
$$= \frac{3}{2}\int_0^1 x \cdot y^2\Big|_{x^2}^{\sqrt{x}}\,dx$$
$$= \frac{3}{2}\int_0^1 x(x - x^4)\,dx$$
$$= \frac{3}{2}\int_0^1 x^2 - x^5\,dx$$
$$= \frac{3}{2}\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = \frac{3}{2}\cdot\left[\frac{1}{3} - \frac{1}{6}\right] = \frac{3}{2}\cdot\frac{1}{6} = \frac{1}{4}$$

$y = \sqrt{x}$
$y = x^2$
$x^2 = \sqrt{x}$
$x^4 = x$
$x^4 - x = 0$
$x(x^3 - 1) = 0$
$x = 0$ or $x^3 - 1 = 0$
$x = 1$

If $x = 1 \rightsquigarrow y = 1$

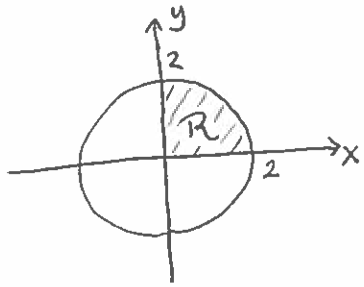
(b) Consider the integral $I = \int_0^1 \int_{2x}^2 2xe^{y^3}\, dy\, dx$. Sketch the region of integration and write I as an iterated integral, complete with limits, with the order of integration reversed. **DO NOT** evaluate the integral.



$$I = \int_0^2 \int_0^{y/2} 2x e^{y^3}\,dx\,dy$$

5. A lamina occupies the region R given by $0 \le y \le \sqrt{4-x^2}$ and $0 \le x \le 2$. The density function is given by $\rho(x,y) = x^2 + y^2$.

(a) Compute the mass of the lamina using polar coordinates.



$$\iint_R \rho(x,y)\,dA = \int_0^{\frac{\pi}{2}}\int_0^2 r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^2 r^3\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^2 r^3\,dr$$

$$= \frac{\pi}{2} \cdot \left.\frac{r^4}{4}\right|_0^2$$

$$= \frac{\pi}{2} \cdot \frac{16}{4}$$

$$= \frac{\pi}{2} \cdot 4$$

$$= 2\pi$$

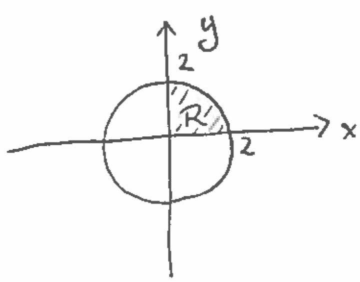
$0 \le y \le \sqrt{4-x^2}$

$0 \le y^2 \le 4 - x^2$

$0 \le x^2 + y^2 \le 4$

w.r.t $0 \le x \le 2$

(b) Write an iterated integral, complete with limits to compute the moment of inertia of the lamina about the y-axis. **Do not** evaluate the integral.



$$M_y = \frac{1}{m}\iint y\,\rho(x,y)\,dA$$

$$= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 r\sin\theta \cdot r^2 \cdot r\,dr\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 r^4 \sin\theta\,dr\,d\theta$$