Solutions

Math 397 Spring 2016 Exam I Name: Caleb McWhorter

Do all your work on this exam. Correct answers should be supported by your calculations and reasoning where appropriate. Please **silence** and **put away** all cell phones and similar devices, including earbuds. **No calculators are allowed.** You need not simplify.

Problem	Points	Score
1	10	
2	12	
3	10	
4	14	
5	14	
Total	60	

1. (10 points) (a) Find the center and radius of the sphere with equation $x^2-2x+y^2+6y+z^2+8z+17=0$.

$$x^2-2x+1+y^2+6y+9+z^2+8z+16+17=1+9+16$$

$$(x^2-2x+1)+(y^2+6y+9)+(z^2+8z+16)=9$$

$$(x-1)^2+(y+3)^2+(z+4)^2=9$$

Radius: 3

Center: $(1,-3,-4)$

(b) Find the intersection of the sphere in part (a) with the yz-plane.

In yz-plane, $x=0$

$$(0-1)^2+(y+3)^2+(z+4)^2=9$$

$$1+(y+3)^2+(z+4)^2=9$$

$$(y+3)^2+(z+4)^2=8$$

Intersection is circle in yz-plane

Center: $(0,-3,-4)$

Radius: $\sqrt{8}=2\sqrt{2}$

(c) Find the intersection of the sphere in part (a) with the xz-plane. Explain how you could find this answer just by looking at the equation of the sphere.

In xz-plane, $y=0$

$$(x-1)^2+(0+3)^2+(z+4)^2=9$$

$$(x-1)^2+9+(z+4)^2=9$$

$$(x-1)^2+(z+4)^2=0$$

So $x=1$, $z=-4$

So intersection is point: $(1,0,-4)$

The sphere is 3 units away from the xz-plane, so the sphere should 'just barely' touch the plane on its 'side'.

2. (12 points) (a) Find the point of intersection of the following pair of lines.

$$L_1: \begin{array}{l} x = 3+t \\ y = 6+2t \\ z = 2-t \end{array} \quad -\infty < t < \infty \text{ and } \quad L_2: \begin{array}{l} x = -4+2s \\ 7 = 7-s \\ z = 12-3s \end{array} \quad -\infty < s < \infty$$

$$\left.\begin{array}{l} 3+t = -4+2s \\ 6+2t = 7-s \\ 2-t = 12-3s \end{array}\right\} \longrightarrow \begin{array}{l} 3+t = -4+2s \\ 6+2t = 7-s \end{array} \rightarrow \begin{array}{l} t-2s = -7 \\ 2t+s = 1 \end{array}$$

$$\begin{array}{rl} & t - 2s = -7 \\ + & 4t + 2s = 2 \\ \hline & 5t + 0 = -5 \\ & t = -1 \end{array}$$

Then $s = \ldots$

$2t+s=1$
$2(-1)+s=1$
$-2+s=1$
$s=3$

$2-t \stackrel{?}{=} 12-3s$
$2--1 \stackrel{?}{=} 12-3(3)$
$3 \stackrel{?}{=} 12-9$
$3 \stackrel{\checkmark}{=} 3$

$x = 3+t = 2$
$y = 6+2t = 4$
$z = 2-t = 3$

$(2,4,3)$

(b) Find the angle between the lines in part (a).

$L_1: \langle 1,2,-1\rangle \stackrel{def}{=} \vec{u}$

$L_2: \langle 2,-1,-3\rangle \stackrel{def}{=} \vec{v}$

$\vec{u}\cdot\vec{v} = 1(2) + 2(-1) + (-3)(-1)$
$= 2 - 2 + 3$
$= 3$

$|\vec{u}| = \sqrt{1^2+2^2+1^2} = \sqrt{1+4+1} = \sqrt{6}$

$|\vec{v}| = \sqrt{2^2+1^2+3^2} = \sqrt{4+1+9} = \sqrt{14}$

$\vec{u}\cdot\vec{v} = \cos\theta \cdot |\vec{u}|\,|\vec{v}|$

$3 = \sqrt{6}\sqrt{14}\cos\theta$

$\cos\theta = \dfrac{3}{\sqrt{6}\sqrt{14}}$

$\theta = \cos^{-1}\left(\dfrac{3}{\sqrt{6}\sqrt{14}}\right)$

3. (10 points) (a) Given any two vectors **a** and **b**, **explain** why $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

The vector $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ & $\vec{b}$

Dotting any perpendicular vectors always gives 0.

So naturally $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$

(b) Find parametric equations for the line containing the point (3,−6, 4) which is perpendicular to the plane $2x - 3y + 7z = 10$.

To be perpendicular to plane, our line must have direction parallel to its normal : $\langle 2, -3, 7 \rangle$

$$r(t) = \langle 2, -3, 7 \rangle t + \langle 3, -6, 4 \rangle$$
$$= \langle 2t + 3, \ -3t - 6, 7t + 4 \rangle$$

$$\begin{cases} x = 2t + 3 \\ y = -3t - 6 \\ z = 7t + 4 \end{cases}$$

4. (14 points) (a) Find the area of the triangle with vertices at the three points $A(1, 3, -2)$, $B(-2, 3, 1)$ and $C(2, 4, 0)$.

$AB = \langle 1 - {}^-2, 3-3, -2-1 \rangle = \langle 3, 0, -3 \rangle$

$CB = \langle 2 - {}^-2, 4-3, 0-1 \rangle = \langle 4, 1, -1 \rangle$

$$\vec{AB} \times \vec{CB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -3 \\ 4 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(0 - {}^-3) - \hat{j}(-3 - {}^-12) + \hat{k}(3 - 0)$$

$$= 3\hat{i} - 9\hat{j} + 3\hat{k}$$

$$= \langle 3, -9, 3 \rangle$$

$$|\langle 3, -9, 3 \rangle| = |3\langle 1, -3, 1 \rangle| = 3|\langle 1, -3, 1 \rangle| = 3\sqrt{1^2 + 3^2 + 1^2}$$

$$= 3\sqrt{1+9+1}$$

$$= 3\sqrt{11}.$$

$$\boxed{\frac{3\sqrt{11}}{2}}$$

(b) Find the equation of the plane containing the three points in part (a).

A possible normal is $\langle 3, -9, 3 \rangle$ but the vector $\langle 1, -3, 1 \rangle$ is 'nicer'.

$$\langle 1, -3, 1 \rangle \cdot \langle x-2, y-4, z-0 \rangle = 0$$

$$(x-2) - 3(y-4) + (z-0) = 0$$

$$x - 2 - 3y + 12 + z = 0$$

$$x - 3y + z = -10$$

5. (14 points) Consider two parallel planes $x+2y+3z=12$ and $x+2y+3z=14$. The first plane contains the point $P(2, 2, 2)$ while the second plane contains the point $Q(5, 3, 1)$.

(a) Find the vector projection of the vector **PQ** onto the normal vector $\mathbf{n} = \langle 1, 2, 3 \rangle$.

PQ has no given direction, assume terminal point Q.

$\vec{PQ} = \langle 5-2, 3-2, 1-2 \rangle = \langle 3, 1, -1 \rangle$

$\text{proj}_n PQ = \frac{n \cdot PQ}{n \cdot n} \vec{n} = \frac{2}{14} \langle 1,2,3 \rangle = \frac{1}{7} \langle 1,2,3 \rangle = \langle 1/7, 2/7, 3/7 \rangle$

$n \cdot PQ = 1(3) + 2(1) + 3(-1)$
$= 3 + 2 - 3$
$= 2$

$n \cdot n = 1^2 + 2^2 + 3^2$
$= 1 + 4 + 9$
$= 14$

(b) Find $comp_{\mathbf{n}}\mathbf{PQ}$, the scalar projection of PQ onto n.

$$\begin{aligned}
\text{comp}_{\vec{n}} \vec{PQ} &= |\text{proj}_n PQ| \\
&= |1/7 \langle 1,2,3 \rangle| \\
&= \tfrac{1}{7} |\langle 1,2,3 \rangle| \\
&= \tfrac{1}{7} \sqrt{1^2 + 2^2 + 3^2} \\
&= \tfrac{1}{7} \sqrt{1 + 4 + 9} \\
&= \frac{\sqrt{14}}{7}
\end{aligned}$$

(c) What does your answer to part (b) tell you geometrically? (Hint: Draw a picture and try to visualize situation.)

This is the distance between the two planes.