Solutions

Math 397 Spring 2016 **Exam III** Name: Caleb McWhorter

Do all your work on this exam. Correct answers should be supported by your calculations and reasoning where appropriate. Please **silence** and **put away** all cell phones and similar devices, including earbuds. **No calculators are allowed.** Unless directed to do so, you do not have to simplify.

Problem	Points	Score
1	15	
2	6	
3	6	
4	12	
5	15	
6	8	
7	8	
Total	70	

1. (15 points) **Find**, and **classify** as relative maximums, relative minimums or saddle points, all **critical points** of $f(x,y) = 2x^2 + y^2 - xy^2$.

$$\begin{cases} f_x = 4x - y^2 = 0 \\ f_y = 2y - 2xy = 0 \end{cases}$$

$$\begin{cases} 4x = y^2 \\ y - xy = 0 \end{cases}$$

$$\begin{cases} 4x = y^2 \\ y(1-x) = 0 \end{cases}$$

$y = 0$ or $1 - x = 0$, $x = 1$

If $y = 0$:
$4x = y^2$
$4x = 0$
$x = 0$
$(0,0)$

If $x = 1$:
$4x = y^2$
$4 = y^2$
$y = \pm 2$
$(1,2)$
$(1,-2)$

$f_{xx} = 4$
$f_{yy} = 2(1-x)$
$f_{xy} = f_{yx} = 2y$

$$d = \begin{vmatrix} 4 & 2y \\ 2y & 2(1-x) \end{vmatrix} = 8(1-x) - 4y^2$$

At $(0,0)$:
$d = 8(1-0) - 0 = 8 > 0$
$f_{xx} = 4 > 0$
So $(0,0)$ is a minimum.

At $(1,2)$:
$d = 8(1-1) - 4(2^2) = -16 < 0$
So $(1,2)$ is a saddle

At $(1,-2)$:
$d = 8(1-1) - 4(-2)^2 = -16 < 0$
So $(1,-2)$ is a saddle

2. (6 points) Let $z = f(x,y) = 3xe^y$, $x = 5t^4$ and $y = 2t+1$.

(a) Write down the **chain rule formula** for $\frac{dz}{dt}$ in this context.

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$$

(b) Use your formula from part (a) to find $\frac{dz}{dt}$. Write your answer in terms of t but **do not simplify**.

$Z_x = 3e^y = 3e^{2t+1}$

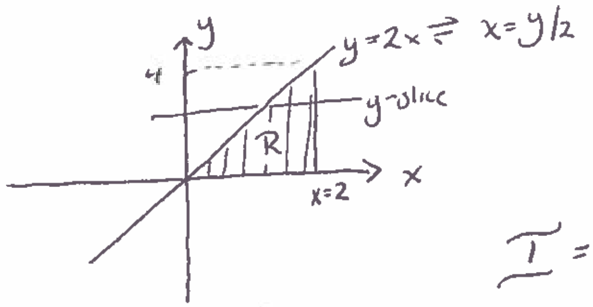
$Z_y = 3xe^y = 3(5t^4)e^{2t+1}$

$X_t = 20t^3$

$y_t = 2$

$$\frac{dz}{dt} = 3e^{2t+1} \cdot 20t^3 + 3(5t^4)e^{2t+1} \cdot 2$$

3. (6 points) Consider the integral $I = \int_0^2 \int_0^{2x} f(x,y)\,dy\,dx$. **Sketch** the region of integration in the xy-plane and write I as an iterated integral with the order of integration reversed. **Do not evaluate the integral.**



$$I = \int_0^4 \int_{y/2}^{2} f(x,y)\,dx\,dy$$

If $x=2$:

$y=2x$

$y=2(2)$

$y=4$

4. (12 points) Consider the function $f(x,y,z) = 3x^2z - yz^2$ and the point $P(1,-2,2)$.

(a) Find ∇f and $\nabla f(P)$.

$$\nabla f = \langle 6xz, -z^2, 3x^2 - 2yz \rangle$$

$$\nabla f(1,-2,2) = \langle 6(1)2, -2^2, 3(1^2) - 2(-2)2 \rangle$$
$$= \langle 12, -4, 3+8 \rangle$$
$$= \langle 12, -4, 11 \rangle$$

(b) Find the directional derivative $(D_{\mathbf{u}}f)_P$ where $\mathbf{u} = \frac{1}{\sqrt{6}}\langle -1, 2, 1 \rangle$.

$|\vec{u}| = \frac{1}{\sqrt{6}}\sqrt{1^2 + 2^2 + 1^2} = \frac{\sqrt{6}}{\sqrt{6}} = 1$ So already unit vector

$$D_u f(p) = \nabla f(1,-2,2) \cdot \vec{u} = \langle 12, -4, 11 \rangle \cdot \frac{\langle -1, 2, 1 \rangle}{\sqrt{6}}$$
$$= \frac{-12 - 8 + 11}{\sqrt{6}}$$
$$= -9/\sqrt{6}$$

(c) Write down the equation of the **plane tangent** to the surface $3x^2z - yz^2 = 14$ at the point P.

$$\underbrace{3x^2z - yz^2 - 14}_{f(x,y,z)-14} = 0$$

$$\nabla(f(x,y,z) - 14) = \nabla f = \langle 6xz, -z^2, 3x^2 - 2yz \rangle \big|_{(1,-2,2)} = \langle 12, -4, 11 \rangle$$
$$\vec{\nabla f} \cdot \vec{vP} = 0$$
$$\langle 12, -4, 11 \rangle \cdot \langle x-1, y+2, z-2 \rangle = 0$$
$$12(x-1) - 4(y+2) + 11(z-2) = 0$$

(d) In what direction does the function $f(x,y,z)$ above **decrease** most rapidly at P?

Increases most rapidly in direction $\nabla f(p) = \langle 12, -4, 11 \rangle$

So decreases most rapidly in direction $-\nabla f(p) = \langle -12, 4, -11 \rangle$

5. (15 points) Use the method of Lagrange multipliers to find maximum and minimum values of $f(x,y) = 6x^2y$ on the ellipse $3x^2 + 2y^2 = 18$.

$$\nabla f = \langle 12xy, 6x^2 \rangle$$

$$\underbrace{3x^2 + 2y^2 - 18}_{g(x,y)} = 0$$

$$\nabla g = \langle 6x, 4y \rangle$$

$$\begin{cases} 12xy = \lambda 6x \\ 6x^2 = \lambda 4y \end{cases}$$

$$3x^2 + 2y^2 = 18$$

$$\begin{cases} 2xy = x\lambda \\ 3x^2 = 2y\lambda \end{cases}$$

$$3x^2 + 2y^2 = 18$$

If $y = 0 \rightarrow 6x^2 = 0 \rightarrow x = 0$

$(0,0)$ but $3 \cdot 0^2 + 2 \cdot 0^2 \neq 18$

$\Rightarrow\Leftarrow$

If $x = 0 \rightarrow 2y\lambda = 0$ so

i) $\lambda = 0 \rightarrow 0 + 2y^2 = 18$

$y^2 = 9$

$y = \pm\sqrt{3}$

* Giving $f(0,-) = 0 \longrightarrow$ which is clearly not a solution.

ii) $y = 0$: $(0,0)$, again no!?

If $x \neq 0, y \neq 0$

$$\begin{cases} 2y = \lambda \\ 3x^2 = 2y\lambda \end{cases}$$

$$3x^2 + 2y^2 = 18$$

$$3x^2 = 2y\lambda$$

$$3x^2 = 2y \cdot 2y$$

$$3x^2 = 4y^2$$

Then...

$$3x^2 + 2y^2 = 18$$

$$4y^2 + 2y^2 = 18$$

$$6y^2 = 18$$

$$y^2 = 3$$

$$y = \pm\sqrt{3}$$

Then...

$$3x^2 + 2y^2 = 18$$

$$3x^2 + 2(3) = 18$$

$$3x^2 + 6 = 18$$

$$3x^2 = 12$$

$$x^2 = 4$$

$$x = \pm 2$$

$(2, \sqrt{3})$, $(-2, \sqrt{3})$

$(-2, \sqrt{3})$, $(-2, -\sqrt{3})$

$f(\pm 2, \sqrt{3}) = 6(\pm 2)^2 \cdot \sqrt{3} = 24\sqrt{3}$

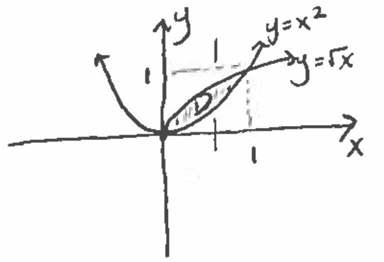
$f(\pm 2, -\sqrt{3}) = 6(\pm 2)^2 \cdot -\sqrt{3} = -24\sqrt{3}$

So $(2, \sqrt{3})$, $(-2, \sqrt{3})$ are max. with value $24\sqrt{3}$

&

$(2, -\sqrt{3})$, $(-2, -\sqrt{3})$ are min. with value $-24\sqrt{3}$

6. (8 points) **Evaluate** $I = \iint_D 3xy\,dA$ where D is the finite region bounded by the curves $y = x^2$ and $y = \sqrt{x}$. Include a **picture** of the region on integration in the xy-plane.



$$\iint_D 3xy\,dA = \int_0^1 \int_{x^2}^{\sqrt{x}} 3xy\,dy\,dx$$

$$= \int_0^1 3x \int_{x^2}^{\sqrt{x}} y\,dy\,dx$$

$$= \int_0^1 3x \cdot \frac{y^2}{2}\Big|_{x^2}^{\sqrt{x}}\,dx$$

$$= \frac{3}{2}\int_0^1 x \cdot y^2\Big|_{x^2}^{\sqrt{x}}\,dx$$

$$= \frac{3}{2}\int_0^1 x(x - x^4)\,dx$$

$$= \frac{3}{2}\int_0^1 x^2 - x^5\,dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = \frac{3}{2}\left[\frac{1}{3} - \frac{1}{6}\right] = \frac{3}{2}\cdot\frac{1}{6} = \frac{1}{4}$$

$y = \sqrt{x}$
$y = x^2$

$x^2 = \sqrt{x}$

$x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$ or $x^3 - 1 = 0$

$x = 1$

If $x = 1 \rightarrow y = 1$

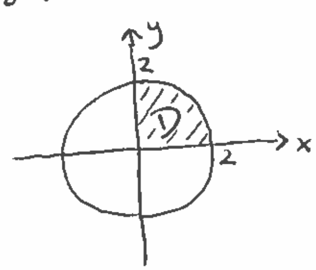
7. (8 points) An integral is given by $I = \iint_D y\,dA$, where D is the region given by $0 \le y \le \sqrt{4 - x^2}$ and $0 \le x \le 2$. **Sketch** the region of integration D in the xy-plane. Then write I as an iterated integral using **polar coordinates** and **evaluate**.

$0 \le y \le \sqrt{4 - x^2}$

$0 \le y^2 \le 4 - x^2$

$0 \le x^2 + y^2 \le 4$

& want $0 \le x \le 2$



$$\iint_D y\,dA = \int_0^{\frac{\pi}{2}} \int_0^2 r\sin\theta \cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 r^2 \sin\theta\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \int_0^2 r^2\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\,d\theta \cdot \int_0^2 r^2\,dr$$

$$= -\cos\theta\Big|_0^{\frac{\pi}{2}} \cdot \frac{r^3}{3}\Big|_0^2$$

$$= (-0 - -1)\cdot\left(\frac{8}{3} - 0\right)$$

$$= 1 \cdot 8/3$$

$$= 8/3$$