Solutions

MAT 397 CAL III Section M007 Fall 2015

TEST 1

Your Name (please PRINT): Caleb McWhorter

Student ID Number: ________

INSTRUCTIONS

- Fill in the above items.
- There is a total of 5 problems, for a maximum possible total value of 100 points. **Make sure you have all 6 test pages (this cover page + 5 test pages).** You are responsible to check that your test booklet has all 6 pages. Alert a proctor if your copy is missing any pages.
- **Show all your work.** Only minimal credit will be given for answers without supporting work.
- **Write your answer in the box** at the bottom of pages 2-6.
- **Use the back of test pages if additional space is needed**, and for scratch paper.
- **No calculators or other electronic devices; no outside notes; no outside tables** are allowed on this exam. Any use of calculators or electronic devices, or outside notes is a violation of the Academic Integrity Policy.

Do not write below this line

Pb. #	Max Points	Your Score
1	24	
2	14	
3	24	
4	20	
5	18	
Total	100	

1. (24 pts) Consider the two vectors

$$\mathbf{a} = \langle 1, -1, 2 \rangle, \qquad \text{and} \qquad \mathbf{b} = \langle -1, -1, 4 \rangle.$$

(a) Find the **angle** between **a** and **b**.

$\vec{a} \cdot \vec{b} = 1(-1) + (-1)(-1) + 2(4)$
$= -1 + 1 + 8 = 8$

$|\vec{a}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$
$|\vec{b}| = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{1+1+16} = \sqrt{18}$

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$
$8 = \sqrt{6}\sqrt{18} \cos\theta$
$8 = 6\sqrt{3} \cos\theta$
$\cos\theta = \frac{4}{3\sqrt{3}}$
$\theta = \cos^{-1}\left(4/3\sqrt{3}\right)$
or
$\cos^{-1}\left(4/\sqrt{27}\right)$

(b) Find a vector that is **perpendicular** to both **a** and **b**.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ -1 & -1 & 4 \end{vmatrix}$$

$= \hat{i}(-4 - -2) - \hat{j}(4 - -2) + \hat{k}(-1 - 1)$
$= -2\hat{i} - 6\hat{j} - 2\hat{k}$
$= \langle -2, -6, -2 \rangle$ or $\langle 2, 6, 2 \rangle$ or $\boxed{\langle 1, 3, 1 \rangle}$

(c) Find the **area of the parallelogram** determined by **a** and **b**.

$A = |\vec{a} \times \vec{b}| = |\langle -2, -6, -2 \rangle|$
$= \sqrt{2^2 + 6^2 + 2^2}$
$= \sqrt{4 + 36 + 4}$
$= \sqrt{44}$
$= 2\sqrt{11}$

Answer for part (a):	$\cos^{-1}\left(4/\sqrt{27}\right)$
Answer for part (b):	$\langle -2, -6, -2 \rangle$
Answer for part (c):	$\sqrt{44}$

2. (14 pts) Find the **scalar** equation of the **plane** that goes through the point $(-1, 2, -2)$ and is **parallel** to the plane with equation $2x - y + 3z = \sqrt{6}$.

To be // to the given plane, our plane must have // normal. So $\langle 2, -1, 3 \rangle$ will work.

$$\langle 2, -1, 3 \rangle \cdot \langle x - {}^-1, y - 2, z - {}^-2 \rangle = 0$$

$$2(x+1) - (y-2) + 3(z+2) = 0$$

$$2x - y + 3z + 2 + 2 + 6 = 0$$

$$2x - y + 3z = -10$$

Answer: $2x - y + 3z = -10$

3. (24 pts) The points P_1, P_2, P_3 and P_4 are given by the x, y, z coordinates $(1,0,0)$, $(0,0,1)$, $(0,2,0)$ and $(2,0,2)$ respectively.

P_1 P_2 P_3 P_4

(a) Write parametric equations of the line L_1 connecting P_1 and P_2.

$$\vec{P_1P_2} = \langle 1-0, 0-0, 0-1 \rangle = \langle 1, 0, -1 \rangle$$

$$r(t) = \langle 1, 0, -1 \rangle t + \langle 1, 0, 0 \rangle = \langle t+1, 0, -t \rangle$$

$$\begin{cases} x = t+1 \\ y = 0 \\ z = -t \end{cases}$$

(b) Write parametric equations of the line L_2 connecting P_3 and P_4.

$$\vec{P_3P_4} = \langle 0-2, 2-0, 0-2 \rangle = \langle -2, 2, -2 \rangle \text{ or } \langle 2, -2, 2 \rangle \text{ or } \langle 1, -1, 1 \rangle$$

$$r(t) = \langle 1, -1, 1 \rangle t + \langle 0, 2, 0 \rangle = \langle t, 2-t, t \rangle$$

$$\begin{cases} x = t \\ y = 2-t \\ z = t \end{cases}$$

(c) Determine whether L_1 and L_2 intersect, are parallel or skew and explain why.

$\langle 1, 0, -1 \rangle \not\parallel \langle 1, -1, 1 \rangle$ so they are not parallel.

If they intersect:

$$\begin{aligned} t+1 &= s \\ 0 &= 2-s \\ -t &= s \end{aligned} \longrightarrow s = 2 \quad \text{so } t = -2$$

$$\begin{aligned} t+1 &\overset{?}{=} s \\ -2+1 &\overset{?}{=} 2 \\ -1 &\neq 2 \end{aligned}$$

So they do not intersect. Then these lines are skew.

Answer for part (a):	$x = t+1$; $y = \cancel{2+t}\ 0$; $z = -t$
Answer for part (b):	$x = t$; $y = 2-t$; $z = t$
Answer for part (c):	Skew

4. (20 pts) A curve is given by the following parametric equation:

$$\vec{r}(t) = 2\cos(2t)\hat{\mathbf{i}} + 2\sin(2t)\hat{\mathbf{j}} + 3t\hat{\mathbf{k}}$$

(a) Find its unit tangent vector $\hat{\mathbf{T}}(t)$.

$r(t) = \langle 2\cos 2t, 2\sin 2t, 3t \rangle$

$r'(t) = \langle -4\sin 2t, 4\cos 2t, 3 \rangle$

$|r'(t)| = \sqrt{(-4\sin 2t)^2 + (4\cos 2t)^2 + 3^2}$

$= \sqrt{16\sin^2 2t + 16\cos^2 2t + 9}$

$= \sqrt{16 + 9}$

$= \sqrt{25}$

$= 5$

So $T_u(t) = \dfrac{T(t)}{|T(t)|}$

$= \dfrac{r'(t)}{|r'(t)|}$

$= \left\langle -\frac{4}{5}\sin 2t, \frac{4}{5}\cos 2t, 3/5 \right\rangle$

(b) Find its principal unit normal vector $\hat{\mathbf{N}}(t)$.

$T(t) = \langle -4\sin 2t, 4\cos 2t, 3 \rangle$

$T'(t) = \langle -8\cos 2t, -8\sin 2t, 0 \rangle$

$|T'(t)| = \sqrt{(-8\cos 2t)^2 + (-8\sin 2t)^2 + 0^2}$

$= \sqrt{64\cos^2 2t + 64\sin^2 2t}$

$= \sqrt{64}$

$= 8$

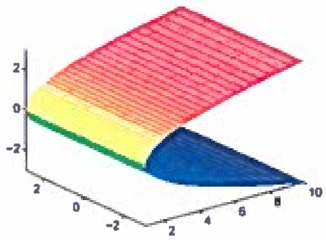
$N_u(t) = \dfrac{T'(t)}{|T'(t)|}$

$= \langle -\cos 2t, -\sin 2t, 0 \rangle$

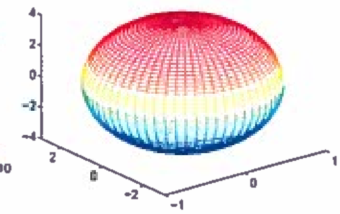
Answer for part (a):	$\langle -4/5\sin 2t, 4/5\cos 2t, 3/5 \rangle$
Answer for part (b):	$\langle -\cos 2t, -\sin 2t, 0 \rangle$

5. (18 pts) Match the following functions with the given graphs and explain why.

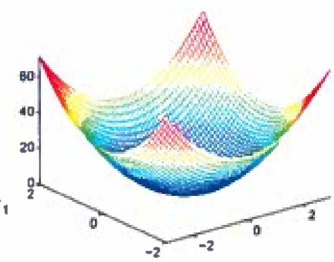
$$(1)\; x^2 + \frac{y^2}{9} + \frac{z^2}{16} = 1 \qquad (2)\; z = 4x^2 + 9y^2 \qquad (3)\; x = z^2 + 1$$



(a)



(b)



(c)

(1) Notice for any fixed value of x, y, z not sufficiently large, we obtain an ellipse. So level curves in each of the $\hat{x}, \hat{y}, \hat{z}$ directions must be ellipses. Therefore, this must be (b).

(2) For fixed values of z, we obtain an ellipse. The larger the chosen z, the larger the ellipse. Therefore, this must be (c).

(3) For any y, we see a parabola in the xz-plane. Therefore, this must be (a).