Solutions

MAT 397 CAL III Section M005 Spring 2016

TEST 2

Your Name (please PRINT): Caleb McWhorter

Student ID Number: ______________________

INSTRUCTIONS

- Fill in the above items.
- There is a total of 5 problems, for a maximum possible total value of 100 points. **Make sure you have all 6 test pages (this cover page + 5 test pages)**. You are responsible to check that your test booklet has all 6 pages. Alert a proctor if your copy is missing any pages.
- **Show all your work.** Only minimal credit will be given for answers without supporting work.
- **Write your answer in the box** at the bottom of pages 2-6.
- **Use the back of test pages if additional space is needed**, and for scratch paper.
- **No calculators or other electronic devices; no outside notes; no outside tables** are allowed on this exam. Any use of calculators or electronic devices, or outside notes is a violation of the Academic Integrity Policy.

Do not write below this line

Pb. #	Max Points	Your Score
1	20	
2	18	
3	18	
4	24	
5	20	
Total	100	

1. (20 pts) Let $f(x, y) = x^2y^2 - x$.

(a) Find ∇f at $(2, 1)$

$\nabla f = \langle 2xy^2 - 1, 2x^2y \rangle$

$\nabla f(2,1) = \langle 2(2)1^2 - 1, 2(2^2)1 \rangle$

$= \langle 4-1, 2(4) \rangle$

$= \langle 3, 8 \rangle$

(b) Use a linear approximation to find the approximate value of $f(1.9, 1.1)$.

$f(2,1) = 2^2(1^2) - 2$

$= 4 - 2$

$= 2$

The tangent plane is (using (a))

$z = 2 + 3(x-2) + 8(y-1)$

So our approximation is

$z_{approx} = 2 + 3(1.9-2) + 8(1.1-1)$

$= 2 + 3(-.1) + 8(.1)$

$= 2 - .3 + .8$

$= 2.5$

OR

$dz = z_x\,dx + z_y\,dy$

$dz = 3\,dx + 8\,dy$

So we have

$dz = 3(-.1) + 8(.1)$

$= -.3 + .8$

$= .5$

$z_{approx} = z_0 + dz = 2 + .5 = 2.5$

Answer for part (a):	$\langle 3, 8 \rangle$
Answer for part (b):	$2\frac{1}{2} = 5/2$

2. (18 pts) Find the equation of the tangent plane to the given surface at the specified point

$$x^2 + z^2 + yz = e^{xy}, \qquad (1, 0, 2)$$

* Note: (1,0,2) is not on the surface $x^2 + z^2 + yz = e^{xy}$. But we shall ignore this issue.....

$$\Downarrow$$

$$\underbrace{x^2 + z^2 + yz - e^{xy}}_{\overset{\text{def}}{=} F(x,y,z)} = 0$$

$$\nabla F = \langle 2x - ye^{xy},\ z - xe^{xy},\ 2z + y \rangle$$

$$\nabla F(1,0,2) = \langle 2(1) - 0,\ 2 - e^0,\ 2(2) + 0 \rangle$$

$$= \langle 2, 1, 4 \rangle$$

So the tangent plane is...

$$\vec{n} \cdot \vec{rP} = 0$$

$$\langle 2,1,4 \rangle \cdot \langle x-1, y-0, z-2 \rangle = 0$$

$$2(x-1) + 1(y-0) + 4(z-2) = 0$$

$$2x - 2 + y + 4z - 8 = 0$$

$$2x + y + 4z = 10$$

Answer: $2x + y + 4z = 10$

3. (18 pts) Let $w = ue^v$, where $u = xy$ and $v = x/y$. Using the chain rule, compute $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ and express them in terms of only x and y.

$w(u,v) = ue^v$

$u(x,y) = xy$

$v(x,y) = x/y$

So.....

$w_u = e^v \qquad u_x = y \qquad v_x = 1/y$

$w_v = ue^v \qquad u_y = x \qquad v_y = -x/y^2$

$$\frac{\partial w}{\partial x} = w_u u_x + w_v v_x$$
$$= e^v y + ue^v \cdot 1/y$$
$$= e^{x/y} y + x\cancel{y} \cdot e^{x/y} \cdot 1/\cancel{y}$$
$$= y e^{x/y} + x e^{x/y}$$
$$= e^{x/y}(x+y)$$

$$\frac{\partial w}{\partial y} = w_u u_y + w_v v_y$$
$$= e^v x + ue^v \cdot -x/y^2$$
$$= e^{x/y} x + xy e^{x/y} \cdot -x/y^2$$
$$= x e^{x/y} - \frac{x^2}{y} e^{x/y}$$
$$= x e^{x/y}(1 - x/y)$$

4. (24 pts) Consider the function

$$f(x,y) = x^3 - xy^2 - 4x^2 + 3x + x^2y$$

(a) Find the maximum value of the directional derivative $D_{\mathbf{u}}f$ at the point $(1,1)$ as $\mathbf{u}$ varies.

The maximum value occurs in the direction of the gradient at (1,1) with value $|\nabla f(1,1)|$

$\nabla f = \langle 3x^2 - y^2 - 8x + 3 + 2xy, \ -2xy + x^2 \rangle$

$\nabla f(1,1) = \langle 3 - 1 - 8 + 3 + 2, \ -2 + 1 \rangle$

$= \langle -1, -1 \rangle$

So

$|\nabla f(1,1)| = |\langle -1, -1 \rangle|$

$= \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

(b) Find the direction $\mathbf{u}$ in which the maximum occurs and $|\mathbf{u}| = 1$.

As above, the direction is the direction of the gradient - $\langle -1,-1\rangle$ However, $\langle -1,-1\rangle$ does not have length 1. So we want, using (a)

$$\frac{\langle -1, -1 \rangle}{\sqrt{2}} = \langle -1/\sqrt{2}, -1/\sqrt{2} \rangle$$

(c) Find the direction(s) $\mathbf{u}$ for which $D_{\mathbf{u}}f(1,1) = 0$ and $|\mathbf{u}| = 1$.

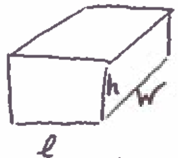
Let $\vec{u} = \langle a, b \rangle$. We want $D_{\vec{u}} f(1,1) = 0$. So

$D_{\vec{u}} f(1,1) = \langle -1, -1 \rangle \cdot \langle a, b \rangle = -a - b = 0 \rightsquigarrow -a = b$. But $|\vec{u}| = 1$

So we have $\sqrt{a^2 + b^2} = 1$. But $\sqrt{a^2 + b^2} = \sqrt{a^2 + (-a)^2} = 1$. Then

$2a^2 = 1 \rightarrow a^2 = 1/2 \rightarrow a = \pm 1/\sqrt{2}$. Then $b = \mp 1/\sqrt{2}$. So we have $\vec{u} = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$ or $\vec{u} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$

Answer for part (a):	$\sqrt{2}$
Answer for part (b):	$\langle -1/\sqrt{2}, -1/\sqrt{2} \rangle$
Answer for part (c):	$\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$ and $\langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$



5. (20 pts) We want to construct a rectangular box. The material used to build the top and bottom cost \$10/ft^2 and the ~~material used to build~~ the sides cost \$5/ft^2. If the box must have a volume of 16 ft^3, determine the dimensions that will minimize the cost to build the box.

Cost: $C(l,w,h) = 10\overset{\to \text{top/bottom}}{(2lw)} + 5\overset{\to \text{front/back}}{(2lh)} + 5\overset{\to \text{left/right}}{(2hw)}$

$= 20lw + 10lh + 10hw$

Know $lwh = 16$ (so $0 \neq l, 0 \neq h, 0 \neq w$)

Via max/min: $lwh = 16 \to h = 16/lw$

$C = 20lw + 10l \cdot \frac{16}{lw} + 10 \cdot \frac{16}{lw} \cdot w$

$C = 20lw + \frac{160}{w} + \frac{160}{l}$

Then ...

$C_l = 20w - \frac{160}{l^2} = 0$

$C_w = 20l - \frac{160}{w^2} = 0$

So $20w = \frac{160}{l^2}$

$w = 8/l^2$

$1/w = l^2/8$

$1/w^2 = l^4/64$

Then $20l - 160 \cdot \frac{1}{w^2} = 0$

$20l - 5l^4/2 = 0$

$l(20 - 5/2\, l^3) = 0$

$l = 0$ or $20 - 5/2\, l^3 = 0$

$5/2\, l^3 = 20$

$l^3 = 8$

$l = 2$

Then $w = 8/l^2 = 8/4 = 2$

$h = \frac{16}{lw} = \frac{16}{2(2)} = 4$

We check this is a min:

$C_{ll} = 320/l^3$

$C_{lw} = 20$

$C_{wl} = 20$

$C_{ww} = 320/w^3$

$\begin{vmatrix} 320/l^3 & 20 \\ 20 & 320/w^3 \end{vmatrix}$

$= \frac{320^2}{(wl)^2} - 400$

$\frac{320^2}{4^2} - 400 > 0$

And $C_{ll} = 320/2^3 > 0$

So this is indeed a minimum value.

OR

Via Lagrange multipliers:

$\nabla C = \lambda \nabla V \quad ; V = lwh$

$\begin{cases} 20w + 10h = \lambda wh \\ 20l + 10h = \lambda lh \\ 10l + 10w = \lambda lw \end{cases}$ $\div wh$, $\div lh$, $\div lw$ $\}\neq 0$

$\begin{cases} 20/h + 10/w = \lambda \\ 20/h + 10/l = \lambda \\ 10/w + 10/l = \lambda \end{cases}$

$\cancel{\frac{20}{h}} + \frac{10}{w} = \lambda = \cancel{\frac{20}{h}} + \frac{10}{l}$

$\frac{10}{w} = \frac{10}{l}$

$w = l$

&

$\frac{20}{h} + \cancel{\frac{10}{l}} = \lambda = \frac{10}{w} + \cancel{\frac{10}{l}}$

$\frac{20}{h} = \frac{10}{w}$

$2w = h$ so $h = 2w = 2l$

$lwh = 16$

$l \cdot l \cdot 2l = 16$

$2l^3 = 16$

$l^3 = 8$

$l = 2$

So $w = l = 2$

$h = 2l = 4$

See other method to verify this is a minimum.

Answer: $l \times w \times h = 2 \times 2 \times 4$ (in ft.)