Solutions

MAT 397 CAL ~~III~~ Section M005 Spring 2016

TEST 3

Your Name (please PRINT): Caleb McWhorter

Student ID Number:

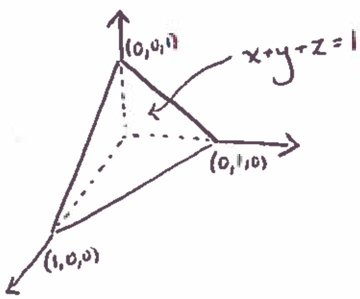
INSTRUCTIONS

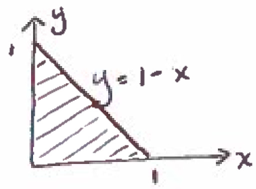
- Fill in the above items.
- There is a total of 5 problems, for a maximum possible total value of 100 points. **Make sure you have all 6 test pages (this cover page + 5 test pages).** You are responsible to check that your test booklet has all 6 pages. Alert a proctor if your copy is missing any pages.
- **Show all your work.** Only minimal credit will be given for answers without supporting work.
- **Write your answer in the box** at the bottom of pages 2-6.
- **Use the back of test pages if additional space is needed,** and for scratch paper.
- **No calculators or other electronic devices; no outside notes; no outside tables** are allowed on this exam. Any use of calculators or electronic devices, or outside notes is a violation of the Academic Integrity Policy.

Do not write below this line

Pb. #	Max Points	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20 pts) Evaluate $\iiint_E y\, dV$, where E is the tetrahedron with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(0,0,0)$.





$$\iiint_E y\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} y(1-x-y)\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} y - xy - y^2\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (1-x)y - y^2\, dy\, dx$$

$$= \int_0^1 \frac{(1-x)}{2}y^2 - \frac{y^3}{3} \Big|_0^{1-x}\, dx$$

$$= \int_0^1 \frac{(1-x)^3}{2} - \frac{(1-x)^3}{3}\, dx$$

$$= \frac{1}{6}\int_0^1 3(1-x)^3 - 2(1-x)^3\, dx$$

$$= \frac{1}{6}\int_0^1 (1-x)^3\, dx$$

$$= \frac{1}{6} \cdot \frac{(1-x)^4}{-4}\Big|_0^1$$

$$= \frac{1}{6} \cdot \left(0 - \frac{1}{-4}\right)$$

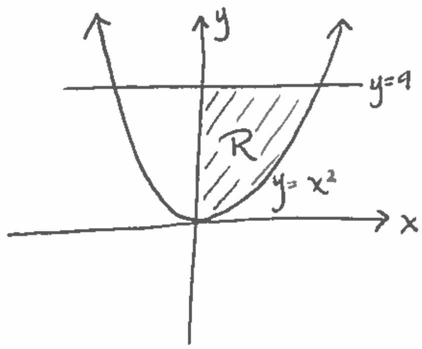
$$= \frac{1}{6} \cdot \frac{1}{4}$$

$$= 1/24$$

Answer:	$1/24$

2. (20 pts) $\iint_R x^3 e^{y^3}\, dA = \int_0^3 \int_{x^2}^9 x^3 e^{y^3}\, dy dx$

(a) Sketch the region R.



(b) Evaluate the integral by first reversing the order of integration.

$$\iint_R x^3 e^{y^3} dA = \int_0^9 \int_0^{\sqrt{y}} x^3 e^{y^3}\, dx\, dy$$

$$= \int_0^9 \frac{x^4}{4} e^{y^3} \Big|_{x=0}^{x=\sqrt{y}} dy$$

$$= \frac{1}{4} \int_0^9 y^2 e^{y^3}\, dy$$

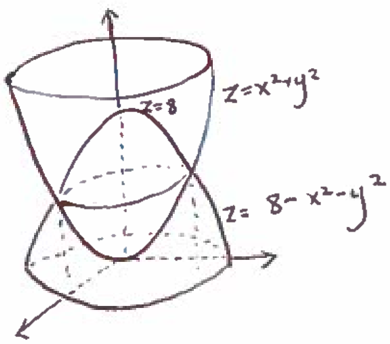
$$= \frac{1}{4} \cdot \frac{1}{3} e^{y^3} \Big|_0^9$$

$$= \frac{1}{12} e^{y^3} \Big|_0^9$$

$$= \frac{1}{12}\left(e^{729} - 1\right)$$

Answer for part (b):	$\frac{1}{12}\left(e^{729} - 1\right)$

3. (20 pts) Find the volume between the two paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$.

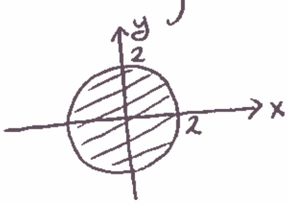


$$\begin{cases} z = x^2+y^2 \\ z = 8-x^2-y^2 \end{cases}$$

$$x^2+y^2 = 8-x^2-y^2$$

$$2x^2+2y^2 = 8$$

$$x^2+y^2 = 4$$



$$V = \iiint_R dV$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^2\int_{r^2}^{8-r^2} r\,dz\,dr$$

$$= 2\pi \int_0^2\int_{r^2}^{8-r^2} r\,dz\,dr$$

$$= 2\pi \int_0^2 rz\Big|_{r^2}^{8-r^2} dr$$

$$= 2\pi \int_0^2 r\left((8-r^2)-r^2\right) dr$$

$$= 2\pi \int_0^2 r(8-2r^2)\,dr$$

$$= 2\pi \int_0^2 8r - 2r^3\,dr$$

$$= 4\pi \int_0^2 4r - r^3\,dr$$

$$= 4\pi\left[2r^2 - \frac{r^4}{4}\right]_0^2$$

$$= 4\pi\left[8-4\right]$$

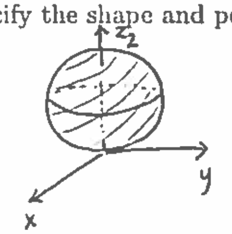
$$= 16\pi$$

Answer: 16π

4. (20 pts) The solid E is bounded by two surfaces $S_1 : \rho = 2\cos\phi$ and $S_2 : \rho = 2$ and for both S_1 and S_2 we have $\phi \in [0, \frac{\pi}{2}]$.

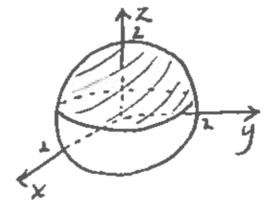
(a) Specify the shape and position of these two surfaces S_1 and S_2.

S_1 :



Whole sphere described

S_2 :



upper half sphere

$S_2 : \rho = 2 \Rightarrow \rho^2 = 4$

$x^2+y^2+z^2=4$

Sphere radius 2 @ (0,0,0)

$S_1 : \rho = 2\cos\phi$

$\rho^2 = 2\rho\cos\phi$

$x^2+y^2+z^2 = 2z$

$x^2+y^2+z^2-2z = 0$

$x^2+y^2+z^2-2z+1 = 1$

$x^2+y^2+(z-1)^2 = 1$

Sphere radius 1 at (0,0,1)

(b) Suppose that the density of this solid E is $\rho(x, y, z) = z$. Find the mass of E.

$$M = \iiint_R \rho(x,y,z)\, dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{2\cos\phi}^{2} \rho\cos\phi \cdot \rho^2\sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{2\cos\phi}^{2} \rho^3 \sin\phi\cos\phi \; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{2}}\int_{2\cos\phi}^{2} \rho^3 \frac{\sin 2\phi}{2}\, d\rho\, d\phi$$

$$= \frac{2\pi}{2}\int_0^{\frac{\pi}{2}}\int_{2\cos\phi}^{2} \rho^3 \sin 2\phi\, d\rho\, d\phi$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{\rho^4}{4}\sin 2\phi \Big|_{2\cos\phi}^{2} d\phi$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} 4\sin 2\phi - 4\cos^4\phi\, \sin 2\phi\, d\phi$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} 4\sin 2\phi\,(1-\cos^4\phi)\, d\phi$$

$$= 4\pi\int_0^{\frac{\pi}{2}} 2\sin\phi\cos\phi\,(1-\cos^4\phi)\, d\phi$$

$$= 8\pi\int_0^{\frac{\pi}{2}} \sin\phi\cos\phi\,(1-\cos^4\phi)\, d\phi$$

$u = \cos\phi$

$du = -\sin\phi\, d\phi$

$$= -8\pi\int_1^0 u(1-u^4)\, du$$

$$= 8\pi\int_0^1 u - u^5\, du$$

$$= 8\pi\left(\frac{u^2}{2} - \frac{u^6}{6}\right)\Big|_0^1 = 8\pi\left(\frac{1}{2}-\frac{1}{6}\right) = \frac{8\pi}{3} = M$$

(c) Set up the triple integrals for finding the center of mass of E. You do not need to evaluate the integrals.

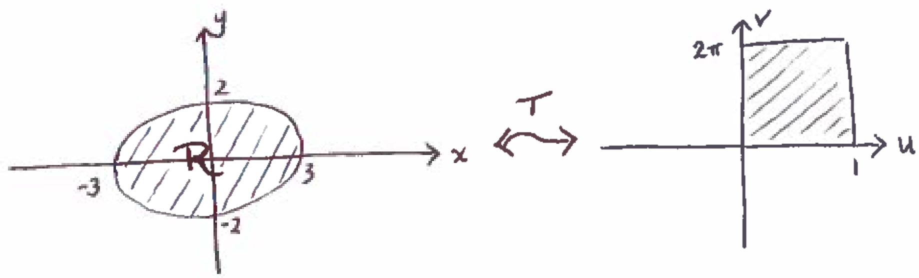
$$\bar{x} = \frac{1}{M}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{2\cos\phi}^{2} \underbrace{\rho\sin\phi\cos\theta}_{x} \cdot \underbrace{\rho\cos\phi}_{z} \cdot \rho^2\sin\phi \; d\rho\, d\phi\, d\theta$$

$$\bar{y} = \frac{1}{M}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{2\cos\phi}^{2} \rho\sin\phi\sin\theta \cdot \rho\cos\phi \cdot \rho^2\sin\phi \; d\rho\, d\phi\, d\theta$$

$$\bar{z} = \frac{1}{M}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{2\cos\phi}^{2} \rho^2\cos^2\phi \cdot \rho^2\sin\phi \; d\rho\, d\phi\, d\theta$$

Answer for part (a):	S_1: Sphere radius 1 at (0,0,1) S_2: Upper 1/2 sphere radius 2 at (0,0,0)
Answer for part (b):	$\frac{8\pi}{3}$
Answer for part (c):	$(\bar{x}, \bar{y}, \bar{z})$ as defined above

5. (20 pts) Use the given transformation $x = 3u\cos v$ and $y = 2u\sin v$ to evaluate the integral $\iint_R \sqrt{\frac{x^2}{9} + \frac{y^2}{4}}\, dA$, where R is enclosed by the ellipse $4x^2 + 9y^2 = 36$. (Note that these are not polar coordinates.)



Region given by

$4x^2 + 9y^2 \le 36$

$4(3u\cos v)^2 + 9(2u\sin v)^2 \le 36$

$36u^2\cos^2 v + 36u^2\sin^2 v \le 36$

$36u^2 \le 36$

$u^2 \le 1$

T: Transformation given by

$x = 3u\cos v$

$y = 2u\sin v$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 3\cos v & -3u\sin v \\ 2\sin v & 2u\cos v \end{vmatrix}$$

$= 6u\cos^2 v + 6u\sin^2 v$

$= 6u(\cos^2 v + \sin^2 v)$

$= 6u$

$$\frac{x^2}{9} + \frac{y^2}{4} = \frac{(3u\cos v)^2}{9} + \frac{(2u\sin v)^2}{4}$$

$= u^2\cos^2 v + u^2\sin^2 v$

$= u^2$

Then $\sqrt{\frac{x^2}{9} + \frac{y^2}{4}} = \sqrt{u^2} = |u| = u$ on $[0,1]$

$$\iint_R \sqrt{\frac{x^2}{9} + \frac{y^2}{4}}\, dA =$$

$$\int_0^{2\pi}\int_0^1 u \cdot 6u \, du\, dv$$

$$6\int_0^{2\pi}\int_0^1 u^2\, du\, dv$$

$$6 \cdot \int_0^{2\pi} dv \cdot \int_0^1 u^2\, du$$

$$\cancel{6}^{\,2} \cdot 2\pi \cdot \frac{1}{\cancel{3}}$$

4π

Answer: 4π