Solutions

Exam 2 **MAT 397 Section 001** **Spring 2016**

This exam has 6 problems on 6 pages. No notes, calculators, or electronic devices of any kind are allowed. **Show all your work!**

Name: Caleb McWhorter

1. (15 points) Decide whether each of the following limits exists. If the limit exists, evaluate it.

a) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^6}{x^2+y^2}$

Along y-axis $(x=0)$: $\lim_{(x,y)\to(0,0)} \frac{0+y^6}{0+y^2} = \lim_{(x,y)\to(0,0)} \frac{y^6}{y^2} = \lim_{(x,y)\to(0,0)} y^4 = 0$

Along x-axis $(y=0)$: $\lim_{(x,y)\to(0,0)} \frac{x^2+0}{x^2+0} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = \lim_{(x,y)\to(0,0)} 1 = 1$

But $1 \neq 0$. Therefore, the limit does not exist.

b) $\lim_{(x,y)\to(0,0)} \frac{x^2+3xy+y^5+12}{x^6+y^6+xy^2+3}$

$\frac{x^2+3xy+y^5+12}{x^6+y^6+xy^2+3}$ is continuous at $(0,0)$. So....

$\lim_{(x,y)\to(0,0)} \frac{x^2+3xy+y^5+12}{x^6+y^6+xy^2+3} = \frac{0+0+0+12}{0+0+0+3} = \frac{12}{3} = 4$

2. (15 points) Suppose the velocity vector of an object at time t is given by $\mathbf{v}(t) = \langle 2t, 4t^3, 3\rangle$.

a) What is the position vector of the object at time 1 if the position vector at time 0 is given by $\mathbf{r}(0) = \langle 1,1,2\rangle$?

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle 2t, 4t^3, 3\rangle\,dt = \langle t^2, t^4, 3t\rangle + \langle C_1, C_2, C_3\rangle$$

$$\vec{r}(0) = \langle 0,0,0\rangle + \langle C_1, C_2, C_3\rangle = \langle C_1, C_2, C_3\rangle.$$

But $\vec{r}(0) = \langle 1,1,2\rangle$. So...

$$\begin{cases} 1 = C_1 \\ 1 = C_2 \\ 2 = C_3 \end{cases}$$

$$\therefore \vec{r}(t) = \langle t^2+1, t^4+1, 3t+2\rangle$$

This gives $\vec{r}(1) = \langle 1^2+1, 1^4+1, 3(1)+2\rangle = \langle 2,2,5\rangle$

b) What is the acceleration vector of the object at time t?

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d}{dt}\langle 2t, 4t^3, 3\rangle = \langle 2, 12t^2, 0\rangle$$

3. (15 points) Let $f(x,y) = x^3 + \sin(xy) + 2e^y$.

a) Compute the linearization $L(x,y)$ of $f(x,y)$ at the point (2,0).

$f(x,y) = x^3 + \sin xy + 2e^y$

$f(2,0) = 2^3 + \sin 0 + 2e^0 = 10$

$\nabla f = \langle 3x^2 + y\cos xy ,\ x\cos xy + 2e^y \rangle$

$\nabla f(2,0) = \langle 3(2^2) + 0,\ 2\cos 0 + 2e^0 \rangle = \langle 12, 4 \rangle$

$L(x,y) \overset{def}{=} f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$\therefore\ L(x,y) = 10 + 12(x-2) + 4(y-0)$ $+10$

b) Find an equation for the tangent plane to the graph of f at the point $(2,0,f(2,0))$.

By the work above, this is simply

$z = 10 + 12(x-2) + 4(y-0)$

$z = 10 + 12x - 24 + 4y$

$12x + 4y - z = 14$

4. (20 points) Suppose the elevation of a mountain range at the point (x, y) is given by the function $f(x, y) = x^2 - xy^3 + 35y$.

a) Find the gradient vector of $f(x, y)$ at the point (3,2).

$$\nabla f = \langle 2x - y^3, -3xy^2 + 35 \rangle$$

$$\nabla f(3,2) = \langle 2(3) - 2^3, -3(3)2^2 + 35 \rangle = \langle -2, -1 \rangle$$

b) Suppose a hiker is standing at the point $(3,2, f(3,2))$. What is the rate of ascent (= slope) if the hiker starts walking in the direction of the unit vector $\mathbf{u} = \frac{1}{5}\langle 3,4 \rangle$?

* Note: $\bar{u}$ is already a unit vector as $|\bar{u}| = \frac{1}{5}\sqrt{3^2+4^2} = \frac{1}{5}\sqrt{25} = 1$

$$D_{\bar{u}} f(3,2) = \vec{\nabla} f(3,2) \cdot \bar{u}$$

$$= \langle -2, -1 \rangle \cdot \frac{\langle 3, 4 \rangle}{5}$$

$$= \frac{-2(3) + -1(4)}{5}$$

$$= \frac{-6-4}{5}$$

$$= \frac{-10}{5}$$

$$= -2$$

5. (20 points) Let $f(x, y) = x^5 + y^5 - 5xy$.

a) Find the critical points of f.

$$\begin{cases} f_x = 5x^4 - 5y = 0 \\ f_y = 5y^4 - 5x = 0 \end{cases}$$

$$\begin{cases} x^4 = y \\ y^4 = x \end{cases}$$

Using first equation in second

$y^4 = x$

$(x^4)^4 = x$

$x^{16} = x$

$x^{16} - x = 0$

But we have

$x^{16} - x = 0$

$x(x^{15} - 1) = 0$

$x = 0$ or $x^{15} - 1 = 0$

$x^{15} = 1$

$x = 1$

Now use fact that $y = x^4$

$x = 0 \rightarrow y = 0^4 = 0$; $(0,0)$

$x = 1 \rightarrow y = 1^4 = 1$; $(1,1)$

b) Determine whether f has a saddle point or a local maximum or minimum at each critical point found in part a.

$f_{xx} = 20x^3$

$f_{yy} = 20y^3$

$f_{xy} = f_{yx} = -5$

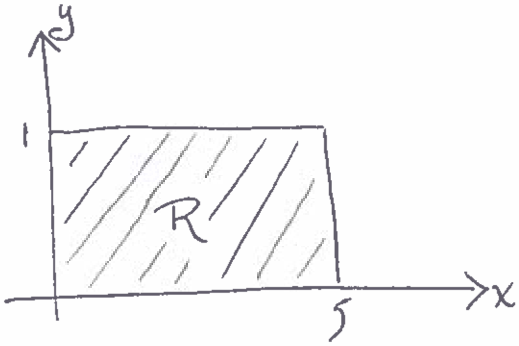
We look at determinant

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 20x^3 & -5 \\ -5 & 20y^3 \end{vmatrix}$$

At $(0,0)$: $\begin{vmatrix} 0 & -5 \\ -5 & 0 \end{vmatrix} = 0 - 25 < 0$. So $(0,0)$ is a saddle point.

At $(1,1)$: $\begin{vmatrix} 20 & -5 \\ -5 & 20 \end{vmatrix} = 400 - 25 > 0$. $f_{xx} = 20 > 0$. So $(1,1)$ is a minimum.

6. (15 points) Use an iterated integral to compute the volume of the solid that lies underneath the surface $z = x + 3xy^2$ and above the square $[0,5] \times [0,1] = \{(x, y) \mid 0 \le x \le 5,\ 0 \le y \le 1\}$.



$$\int_0^5 \int_0^1 x + 3xy^2 \, dy \, dx$$

$$\int_0^5 xy + xy^3 \Big|_{y=0}^{y=1} dx$$

$$\int_0^5 x + x \, dx$$

$$\int_0^5 2x \, dx$$

$$x^2 \Big|_0^5$$

$$25$$

OR =

$$\int_0^1 \int_0^5 x + 3xy^2 \, dx \, dy$$

$$\int_0^1 \frac{x^2}{2} + \frac{3x^2}{2} y^2 \Big|_{x=0}^{x=5} dy$$

$$\int_0^1 \frac{25}{2} + \frac{75}{2} y^2 \, dy$$

$$\frac{25}{2} y + \frac{75}{2} \cdot \frac{y^3}{3} \Big|_{y=0}^{y=1}$$

$$\frac{25}{2} + \frac{75}{2} \cdot \frac{1}{3}$$

$$\frac{25}{2} + \frac{25}{2}$$

$$50/2$$

$$25$$