Solutions

Exam 3 **MAT 397 Section 001** **Spring 2016**

This exam has 6 problems on 6 pages. No notes, calculators, or electronic devices of any kind are allowed. **Show all your work!**

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1. (15 points) Evaluate the double integral $\iint_D 2x\,dA$ where D is the region in $\mathbb{R}^2$ given by the equations $0 \le x \le 1$ and $2x^2 - x \le y \le 2x$.

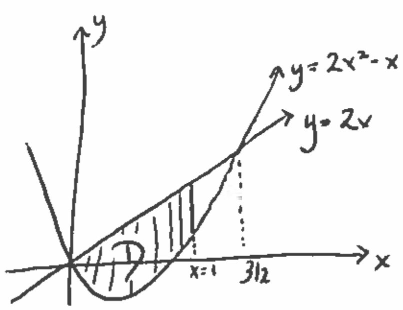
$2x^2 - x = 2x$

$2x^2 - 3x = 0$

$x(2x-3) = 0$

$x=0$ or $2x-3=0$

$x = 3/2$



$$\iint_D 2x\,dA = \int_0^1 \int_{2x^2-x}^{2x} 2x\,dy\,dx$$

$$= \int_0^1 2x \int_{2x^2-x}^{2x} dy\,dx$$

$$= \int_0^1 2x\,(2x - (2x^2 - x))\,dx$$

$$= \int_0^1 2x(3x - 2x^2)\,dx$$

$$= \int_0^1 6x^2 - 4x^3\,dx$$

$$= 2x^3 - x^4 \Big|_0^1$$

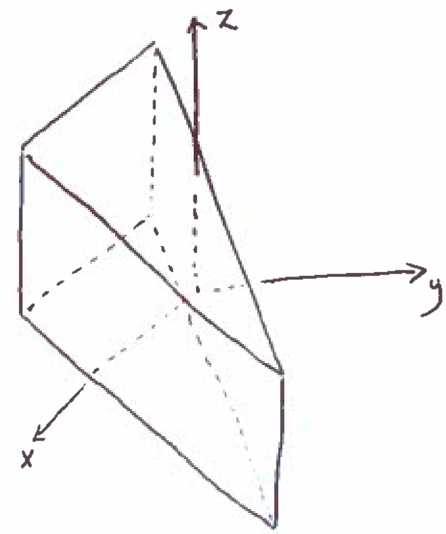
$$= (2-1) - 0$$

$$= 1$$

2. (15 points) Suppose a solid object occupies the region

$$E = \{ (x,y,z) \mid 0 \le x \le \pi/2,\ 0 \le y \le x,\ 0 \le y+z \le 2 \}.$$

Find the total mass of the object if the mass density at (x,y,z) is given by $\rho(x,y,z) = 3 + \cos y$.



$$M = \iiint_E dV$$

$$= \int_0^{\frac{\pi}{2}} \int_0^x \int_0^{2-y} 3 + \cos y \; dz\, dy\, dx$$

$$= \int_0^{\frac{\pi}{2}} \int_0^x (3+\cos y)(2-y)\, dy\, dx$$

$$= \int_0^{\frac{\pi}{2}} \int_0^x 6 - 3y + 2\cos y - y\cos y \; dy\, dx$$

$$= \int_0^{\frac{\pi}{2}} 6y - \frac{3y^2}{2} + 2\sin y - (y\sin y + \cos y) \Big|_0^x dx$$

$$= \int_0^{\frac{\pi}{2}} \left(6x - \frac{3x^2}{2} + 2\sin x - x\sin x - \cos x\right) - -1\, dx$$

$$= \int_0^{\frac{\pi}{2}} 6x - \frac{3x^2}{2} + 2\sin x - x\sin x - \cos x + 1\, dx$$

$$3x^2 - \frac{x^3}{2} - 2\cos x + x\cos x - \sin x - \sin x + x \Big|_0^{\frac{\pi}{2}}$$

$$3x^2 - \frac{x^3}{2} - 2\cos x + x\cos x - 2\sin x + x \Big|_0^{\frac{\pi}{2}}$$

$$\left(3\left(\frac{\pi}{2}\right)^2 - \frac{\left(\frac{\pi}{2}\right)^3}{2} - 0 + 0 - 2 + \frac{\pi}{2}\right) - (-2)$$

$$\frac{3\pi^2}{4} - \frac{\pi^3}{16} \cancel{-2} + \frac{\pi}{2} \cancel{+2}$$

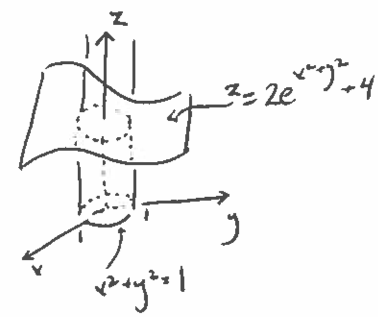
$$\frac{3\pi^2}{4} - \frac{\pi^3}{16} + \frac{\pi}{2}$$

$$\frac{12\pi^2 - \pi^3 + 8\pi}{16}$$

$$\frac{-\pi}{16}\left(\pi^2 - 12\pi - 8\right)$$

3. (20 points) Use cylindrical coordinates to compute the volume $V = \iiint_E 1 dV$ of the region E that lies above xy-plane, inside the cylinder $x^2 + y^2 \leq 1$, and underneath the surface $z = 2e^{x^2+y^2} + 4$.

$$V = \iiint_E dV = \int_0^{2\pi}\int_0^1\int_0^{2e^{r^2}+4} r\,dz\,dr\,d\theta$$



$$= \int_0^{2\pi}\int_0^1 r(2e^{r^2}+4)\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 2re^{r^2} + 4r\,dr$$

$$= 2\pi \int_0^1 2re^{r^2} + 4r\,dr$$

$$= 2\pi \left[e^{r^2} + 2r^2 \right]_0^1$$

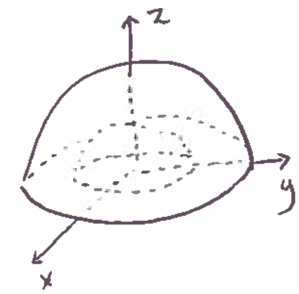
$$= 2\pi \left[(e+2) - (1+0) \right]$$

$$= 2\pi(e+1)$$

4. (15 points) Let E be the region in $\mathbb{R}^3$ which lies above the xy-plane and between the spheres of radius 4 and 5 with centers at the origin. Use spherical coordinates to express the triple integral

$$\iiint_E z\,dV$$

as an iterated integral. Do <u>not</u> evaluate the integral!



$$\iiint_E z\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_4^5 \rho\cos\phi \cdot \rho^2\sin\phi \,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_4^5 \rho^3 \sin\phi\cos\phi \,d\rho\,d\phi\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_4^5 \rho^3 \sin 2\phi \,d\rho\,d\phi\,d\theta$$

5. (20 points) Let $R = \{(x,y) | 0 \le x/2 + y \le 1, 0 \le x + 3y \le 1\}$.

a) Express x and y in terms of the variables $u = x/2 + y$ and $v = x + 3y$.

$$\begin{cases} u = \frac{x}{2} + y \\ v = x + 3y \end{cases}$$

$$\begin{cases} 2u = x + 2y \\ v = x + 3y \end{cases}$$

$$2u - v = -y$$

$$y = v - 2u$$

$$\therefore \quad v = x + 3y$$

$$v = x + 3(v - 2u)$$

$$v = x + 3v - 6u$$

$$x = 6u - 2v$$

$$x = 6u - 2v$$

$$y = v - 2u$$

$$0 \le x + 3y \le 1$$

$$0 \le (6u - 2v) + 3(v - 2u) \le 1$$

$$0 \le 6u - 2v + 3v - 2u \le 1$$

$$0 \le 4u + v \le 1$$

$$0 \le \frac{x}{2} + y \le 1$$

$$0 \le \frac{6u - 2v}{2} + v - 2u \le 1$$

$$0 \le 3u - v + v - 2u \le 1$$

$$0 \le u \le 1$$

b) Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ -2 & 1 \end{vmatrix} = 6 - 4 = 2$$

c) Use parts a) and b) to evaluate the double integral $\iint_R y\,dA$.

$$\iint_R y\,dA = \int_0^1 \int_0^{1-4u} (v - 2u) \cdot 2\,dv\,du = 2\int_0^1 \int_0^{1-4u} v - 2u\,dv\,du =$$

$$2\int_0^1 \frac{v^2}{2} - 2uv \Big|_0^{1-4u} du = 2\int_0^1 \frac{(1-4u)^2}{2} - 2u(1-4u)\,du = 2\int_0^1 \frac{1 + 16u^2 - 8u}{2} - 2u + 8u^2\,du$$

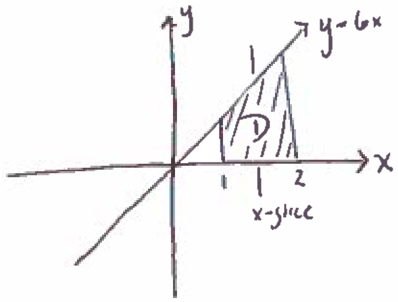
$$= \int_0^1 1 + 16u^2 - 8u - 4u + 16u^2\,du = \int_0^1 32u^2 - 12u + 1\,du = \frac{32u^3}{3} - 6u^2 + u \Big|_0^1 =$$

$$\frac{32}{3} - 6 + 1 = \frac{32}{3} - 5 = \frac{32}{3} - \frac{15}{3} = 17/3$$

6. (15 points) A lamina occupies the region

$$D = \{ (x, y) \mid 1 \le x \le 2,\ 0 \le y \le 6x \}$$

in the xy-plane. Find the center of mass of the lamina if the mass density is constant equal to 1.



$$M = \iint_D \rho(x,y)\,dA = \int_1^2 \int_0^{6x} dy\,dx = \int_1^2 y\Big|_0^{6x} dx$$

$$= \int_1^2 6x\,dx = 3x^2\Big|_1^2 = 3(2^2 - 1^2) = 3(3) = 9$$

$$M_x = \iint_D y\rho(x,y)\,dA = \int_1^2 \int_0^{6x} y\,dy\,dx = \int_1^2 \frac{y^2}{2}\Big|_0^{6x} dx = \int_1^2 18x^2\,dx$$

$$= 6x^3\Big|_1^2 = 6(2^3 - 1^3) = 6(7) = 42$$

$$M_y = \iint_D x\rho(x,y)\,dA = \int_1^2 \int_0^{6x} x\,dy\,dx = \int_1^2 x \cdot 6x\,dx = \int_1^2 6x^2\,dx$$

$$= 2x^3\Big|_1^2 = 2(2^3 - 1^3) = 2(7) = 14$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{14}{9}, \frac{42}{9}\right) = \left(\frac{14}{9}, \frac{14}{3}\right)$$