Math 296: Exam 1
Summer Session II – 2016
07/14/2016
80 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 10 pages (including this cover page) and 8 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
Total:	40	

1. (5 points) Find the limits of the following *sequences*:

$$a_n = \frac{3n^2 - 5n + 7}{5n^2 - 2n + 3}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2 - 5n + 7}{5n^2 - 2n + 3} = \frac{3}{5}$$

$$b_n = n \sin\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

$$c_n = \sqrt[n]{2n}$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \sqrt[n]{2n} = \lim_{n\to\infty} 2^{1/n} \cdot n^{1/n} = 1 \cdot 1 = 1$$

$$d_n = \left(1 + \frac{7}{5n}\right)^{2n/3}$$

$$\begin{aligned}
\lim_{n\to\infty} d_n &= \lim_{n\to\infty} \left(1 + \frac{7}{5n}\right)^{2n/3} \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{5n/7}\right)^{2n/3} \\
&= \lim_{n\to\infty} \left[\left(1 + \frac{1}{5n/7}\right)^{5n/7}\right]^{\frac{7}{5}\cdot\frac{2}{3}} \\
&= e^{\frac{7}{5}\cdot\frac{2}{3}} = e^{14/15} = \sqrt[15]{e^{14}}
\end{aligned}$$

2. (5 points) Determine if the following series converge or diverge. Justify your answer.

$$\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos 0 = 1 \neq 0$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ *diverges by the Divergence Test.*

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

We know that $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ *so that* $\sin\left(\frac{1}{n}\right) \approx \frac{1}{n} - \frac{(1/n)^3}{3!} + \frac{(1/n)^5}{5!} - \cdots$. *For 'large'* n, $\sin(1/n)$ *is positive. Given that* $\sin(1/n)$ *should 'behave' like* $1/n$ *and is positive for large* n, *we use the Limit Comparison Test with* $1/n$.

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 \neq 0$$

Because $\sum \frac{1}{n}$ *diverges by the* p*-test (it is the Harmonic series),* $\sum \sin(1/n)$ *converges by the Limit Comparison Test.*

3. (5 points) Determine if the following series converges or diverges. If the series converges, find the sum. If the series diverges, prove it. $\left[\text{Hint: } \dfrac{1}{n^2+3n+2} = \dfrac{1}{n+1} - \dfrac{1}{n+2}\right]$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

We look at the partial sums:

$$\begin{aligned}
S_1 &= \frac{1}{1} - \frac{1}{2} \\
S_2 &= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3} \\
S_3 &= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4} \\
&\vdots \\
S_m &= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots \left(\cancel{\frac{1}{m}} - \frac{1}{m+1}\right) = 1 - \frac{1}{m+1}
\end{aligned}$$

Then

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \lim_{m\to\infty} S_m = \lim_{m\to\infty} \left(1 - \frac{1}{m+1}\right) = 1 - 0 = 1$$

Therefore, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = 1.$

4. (5 points) Determine if the following series converges or diverges. If the series converges, find the sum. If the series diverges, prove it.

$$\sum_{n=0}^{\infty} \frac{3^{n-2}}{e^{n+3}}$$

$$\sum_{n=0}^{\infty} \frac{3^{n-2}}{e^{n+3}} = \sum_{n=0}^{\infty} \frac{3^n 3^{-2}}{e^n e^3} = \sum_{n=0}^{\infty} \frac{1}{9e^3} \left(\frac{3}{e}\right)^n$$

This series is geometric with $r = 3/e$. *Now* $e \approx 2.718281828\cdots$ *so that* $3/e > 1$. *But then* $|r| = |3/e| > 1$ *so that the series diverges by the Geometric Series Test.*

$$\sum_{n=1}^{\infty} \frac{\pi^n - 2^{n+1}}{3^{2n}}$$

$$\sum_{n=0}^{\infty} \frac{\pi^n - 2^{n+1}}{3^{2n}} = \sum_{n=0}^{\infty} \frac{\pi^n}{3^{2n}} - \frac{2^{n+1}}{3^{2n}} = \sum_{n=0}^{\infty} \frac{\pi^n}{3^{2n}} - \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{2n}}$$

Now $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^n}{3^{2n}} = \sum_{n=0}^{\infty} \left(\frac{\pi}{3^2}\right) = \sum_{n=0}^{\infty} \left(\frac{\pi}{9}\right)^n$. *This series is geometric with* $r = \pi/9$. *But* $\pi \approx 3.14159\ldots$ *so that* $\pi/9 < 1$. *Therefore by the Geometric Series Test, the series converges and sums to* $\dfrac{\pi/9}{1-\pi/9} = \dfrac{\pi}{9-\pi}$.

Now $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{2n}} = \sum_{n=0}^{\infty} \frac{2^n \cdot 2}{3^{2n}} = \sum_{n=0}^{\infty} 2\left(\frac{2}{3^2}\right)^n = \sum_{n=0}^{\infty} 2\left(\frac{2}{9}\right)^n$. *This series is geometric with* $|r| = |2/9| = 2/9 < 1$, *so the series converges by the Geometric Series Test. It converges to* $\dfrac{2 \cdot 2/9}{1-2/9} = \dfrac{4/9}{7/9} = \dfrac{4}{7}$. *Therefore,*

$$\sum_{n=1}^{\infty} \frac{\pi^n - 2^{n+1}}{3^{2n}} = \frac{\pi}{9-\pi} - \frac{4}{7}$$

5. (5 points) Determine if the following series conditionally converges, absolutely converges, or diverges. If the series converges, determine how many terms are needed to add to approximate the sum to three decimal digits of accuracy. If the series diverges, prove it.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

Note that $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$ *and* $\left\{ \frac{1}{\sqrt[3]{n}} \right\}$ *is a decreasing sequence. So see this last part, note that using the fact that* $\sqrt[3]{x}$ *is an increasing function*

$$\begin{aligned} n+1 &> n \\ \sqrt[3]{n+1} &> \sqrt[3]{n} \\ \frac{1}{\sqrt[3]{n+1}} &< \frac{1}{\sqrt[3]{n}} \end{aligned}$$

Alternatively, note that $\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x}}\right) = \frac{-1/3}{x^{4/3}} = \frac{-1}{3\sqrt[3]{x^4}} < 0$. *In any case,* $\left\{ \frac{1}{\sqrt[3]{n}} \right\}$ *meets the two criterion for the Alternating Series Test. Therefore by the Alternating Series Test,* $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ *converges.*

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ *diverges by the* p*-test with* $p = 1/3 \leq 1$. *Therefore,* $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ *converges conditionally.*

To find the sum to 3 decimal places of accuracy, we want $\frac{1}{\sqrt[3]{n}} < \frac{1}{1000} = 0.001$. *So we want* $\sqrt[3]{n} > 1000 = 10^3$ *so that* $n > (10^3)^3 = 10^9$. *So one would need to add the first* $10^9 + 1 = 1,000,000,001$ *terms.*

6. (5 points) Determine if the following series converge or diverge. Be sure to justify your answer.

$$\sum_{n=1}^{\infty} \frac{n^2+n}{n^4+5}$$

$$0 \leq \sum_{n=1}^{\infty} \frac{n^2+n}{n^4+5} \leq \sum_{n=1}^{\infty} \frac{n^2+n^2}{n^4} = \sum_{n=1}^{\infty} \frac{2n^2}{n^4} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the* p*-test. Therefore,* $\sum_{n=1}^{\infty} \frac{n^2+n}{n^4+5}$ *converges by the Comparison Test.*

<u>OR</u>

$$\lim_{n\to\infty} \frac{\frac{n^2+n}{n^4+5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4+n^3}{n^4+5} = 1 < \infty$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the* p*-test. Therefore,* $\sum_{n=1}^{\infty} \frac{n^2+n}{n^4+5}$ *converges by the Limit Comparison Test.*

$$\sum_{n=1}^{\infty} \frac{n+\ln n}{n^2+1}$$

$$\sum_{n=1}^{\infty} \frac{n+\ln n}{n^2+1} \geq \sum_{n=1}^{\infty} \frac{n}{n^2+n^2} = \sum_{n=1}^{\infty} \frac{n}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by the* p*-test. Therefore,* $\sum_{n=1}^{\infty} \frac{n+\ln n}{n^2+1}$ *diverges by the Comparison Test.*

<u>OR</u>

$$\lim_{n\to\infty} \frac{\frac{n+\ln n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n\ln n}{n^2+1} = 1 < \infty$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by the* p*-test. Therefore,* $\sum_{n=1}^{\infty} \frac{n+\ln n}{n^2+1}$ *diverges by the Limit Comparison Test.*

7. (5 points) Determine if the following series converge or diverge. Be sure to be as specific as possible and justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{3^n n^n}$$

$$\lim_{n\to\infty}\left(\left|\frac{(-5)^n}{3^n n^n}\right|\right)^{1/n} = \lim_{n\to\infty}\left(\frac{5^n}{3^n n^n}\right)^{1/n} = \lim_{n\to\infty}\frac{5}{3n} = 0 < 1$$

Therefore, $\sum_{n=1}^{\infty} \frac{(-5)^n}{3^n n^n}$ *converges by the Root Test.*

OR

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-5)^{n+1}}{3^{n+1}(n+1)^{n+1}}\cdot\frac{3^n n^n}{(-5)^n}\right| &= \lim_{n\to\infty}\frac{5^{n+1}}{5^n}\cdot\frac{3^n}{3^{n+1}}\cdot\frac{n^n}{(n+1)^{n+1}}\\ &= \lim_{n\to\infty}\frac{5}{3}\cdot\frac{n^n}{(n+1)^n(n+10}\\ &= \lim_{n\to\infty}\frac{5}{3}\cdot\left(\frac{n}{n+1}\right)^n\frac{1}{n+1}\\ &= \lim_{n\to\infty}\frac{5}{3}\left(1+\frac{1}{n}\right)^{-n}\cdot\frac{1}{n+1} = \frac{5}{3}\cdot e^{-1}\cdot 0 = 0 < 1\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \frac{(-5)^n}{3^n n^n}$ *converges by the Ratio Test.*

$$\sum_{n=1}^{\infty}(-1)^n\frac{2^n\sqrt{n}}{n!}$$

$$\lim_{n\to\infty}\left(\left|(-1)^n\frac{2^n\sqrt{n}}{n!}\right|\right)^{1/n} = \lim_{n\to\infty}\frac{2n^{1/(2n)}}{\sqrt[n]{n!}} = 0 < \infty$$

Therefore, $\sum_{n=1}^{\infty}(-1)^n\frac{2^n\sqrt{n}}{n!}$ *converges by the Root Test.*

OR

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^n\sqrt{n+1}}{(n+1)!}\cdot\frac{n!}{(-1)^n 2^n\sqrt{n}}\right| &= \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{\sqrt{n+1}}{\sqrt{n}}\cdot\frac{n!}{(n+1)!}\\ &= \lim_{n\to\infty}2\sqrt{\frac{n+1}{n}}\cdot\frac{1}{n+1} = 2\cdot\sqrt{1}\cdot 0 = 0 < 1\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty}(-1)^n\frac{2^n\sqrt{n}}{n!}$ *converges by the Ratio Test.*

8. (5 points) Complete the following question by completing each part. Be sure to justify your answer for each part.

(a) Find the first three nonzero terms of the Maclaurin series for $\dfrac{1}{1-x}$.

$$f(x) = \left.\frac{1}{1-x}\right|_{x=0} = 1$$
$$f'(x) = \left.\frac{1}{(1-x)^2}\right|_{x=0} = 1$$
$$f''(x) = \left.\frac{2}{(1-x)^3}\right|_{x=0} = 2$$
$$\frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{2}{2!}x^2 = 1 + x + x^2$$

(b) Use the previous part to find the Maclaurin series for $\dfrac{1}{1-x}$.

$$f^{(n)} = \left.\frac{n!}{(1-x)^{n+1}}\right|_{x=0} = n!$$
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{n!}{n!}x^n = \sum_{n=0}^{\infty} x^n$$

(c) Find the interval and radius of convergence of the power series in (b).

The series is geometric with $r = x$. Therefore by the Geometric Series Test, the series converges if and only if $|r| = |x| < 1$. The interval of convergence is then $(-1, 1)$ with radius of convergence $R = \dfrac{1-(-1)}{2} = \dfrac{2}{2} = 1$.

(d) Use the work in the previous parts to find the sum $\displaystyle\sum_{n=0}^{\infty} \frac{n}{3^n}$.

$$\sum_{n=0}^{\infty} \frac{n}{3^n} = \sum_{n=0}^{\infty} n\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} nx^n$$

if $x = 1/3$. Now

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx}x^n = \sum_{n=0}^{\infty} nx^{n-1}$$

$$x\frac{d}{dx}\left(\frac{1}{1-x}\right) = x\sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} nx^n$$

But $x\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{x}{(1-x)^2}$. *But then*

$$\sum_{n=0}^{\infty} n\left(\frac{1}{3}\right)^n = \frac{1/3}{(1-1/3)^2} = \frac{1/3}{(2/3)^2} = \frac{1/3}{4/9} = \frac{3}{4}$$