Math 296: Exam 2
Summer Session II – 2016
07/28/2016
80 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 10 pages (including this cover page) and 9 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
Total:	45	

1. (5 points) Integrate the following:

$$\int \frac{1+\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx$$

Let $u=\sqrt[3]{x}$. *Then* $du=\dfrac{1}{3\sqrt[3]{x^2}}\,dx=\dfrac{1}{3(\sqrt[3]{x})^2}\,dx=\dfrac{1}{3u^2}\,dx$ *so that* $dx=3u^2\,du$. *But then*

$$\int \frac{1+\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx=\int\frac{1+u}{u-1}\cdot 3u^2\,du=3\int\frac{u^3+u^2}{u-1}\,du$$

Now

$$\begin{array}{r|l} & x^2+2x+2 \\ x-1 & x^3 \;+x^2 \\ & -x^3\;+x^2 \\ \hline & 2x^2 \\ & -2x^2+2x \\ \hline & 2x \\ & -2x+2 \\ \hline & 2 \end{array}$$

Then

$$\begin{aligned}3\int\frac{u^3+u^2}{u-1}\,du&=3\int\left(u^2+2u+2+\frac{2}{u-1}\right)du\\&=u^3+3u^2+6u+6\ln|u-1|+C\\&=x+3\sqrt[3]{x^2}+6\sqrt[3]{x}+6\ln|\sqrt[3]{x}-1|+C\end{aligned}$$

OR

Let $u=\sqrt[3]{x}-1$ *so that* $\sqrt[3]{x}=u+1$ *and* $\sqrt[3]{x^2}=(u+1)^2$. *Then* $du=\dfrac{1}{3\sqrt[3]{x^2}}\,dx=\dfrac{1}{3(\sqrt[3]{x})^2}\,dx=\dfrac{1}{3(u+1)^2}\,dx$ *so that* $dx=3(u+1)^2\,du$. *Now* $\sqrt[3]{x}+1=\sqrt[3]{x}-1+2$. *Then*

$$\begin{aligned}\int\frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx&=\int\frac{\sqrt[3]{x}-1+2}{\sqrt[3]{x}-1}dx\\&=\int\frac{u+2}{u}\cdot 3(u+1)^2\,du\\&=3\int\frac{(u+2)(u^2+2u+1)}{u}\,du\\&=3\int\frac{u^3+4u^2+5u+2}{u}\,du\\&=u^3+6u^2+15u+6\ln|u|+C\\&=(\sqrt[3]{x}-1)^3+6(\sqrt[3]{x}-1)^2+15(\sqrt[3]{x}-1)+6\ln|\sqrt[3]{x}-1|+C\end{aligned}$$

There are even more ways of computing this integral!

2. (5 points) A table of integration gives the following:

$$\int \frac{dx}{x\sqrt{x^n - a^n}} = \frac{2}{n\sqrt{a^n}} \cos^{-1} \sqrt{\frac{a^n}{x^n}}$$

Use this to integrate the following:

$$\int \frac{\ln 2}{\sqrt{2^{5x} - 32}}\, dx$$

$$\int \frac{\ln 2}{\sqrt{2^{5x} - 32}}\, dx = \int \frac{\ln 2}{\sqrt{(2^x)^5 - 2^5}}\, dx = \int \frac{2^x \ln 2}{2^x\sqrt{(2^x)^5 - 2^5}}\, dx$$

Let $u = 2^x$ so that $du = 2^x \ln 2\, dx$. Then using $a = 2$ and $n = 5$ in the formula above, we have

$$\int \frac{du}{u\sqrt{u^5 - 2^5}} = \frac{2}{5\sqrt{2^5}} \cos^{-1}\left(\sqrt{\frac{2^5}{u^5}}\right) + C = \frac{2}{5\sqrt{32}} \cos^{-1}\left(\sqrt{\frac{32}{2^{5x}}}\right) + C$$

3. (5 points) Integrate the following:

$$\int x^3 e^{x^2}\ dx$$

x^2	$\dfrac{e^{x^2}}{2}$
$2x$	xe^{x^2}

$$\int x^3 e^{x^2}\ dx = \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}\ dx = \frac{1}{2}x^2e^{x^2} - \frac{e^{x^2}}{2} + C = \frac{e^{x^2}}{2}(x^2-1) + C$$

$$\int \arctan x\ dx$$

$\arctan x$	x
$\dfrac{1}{1+x^2}$	1

$$\begin{aligned}\int \arctan x\ dx &= x\arctan x - \int \frac{x}{1+x^2}\ dx \\ &= x\arctan x - \frac{1}{2}\ln|1+x^2| + C \\ &= x\arctan x - \ln|\sqrt{1+x^2}| + C \\ &= x\arctan x + \frac{1}{2}\ln\left(\frac{1}{1+x^2}\right) + C\end{aligned}$$

4. (5 points) Integrate the following:

$$\int e^{2x} \sin 3x \; dx$$

u		dv
$\sin(3x)$		e^{2x}
	$+$	
$3\cos(3x)$	$-$	$\frac{e^{2x}}{2}$
$-9\sin(3x)$	$+$	$\frac{e^{2x}}{4}$

$$\int e^{2x} \sin 3x \; dx = \frac{1}{2} e^{2x} \sin(3x) - \frac{3}{4} e^{2x} \cos(3x) - \frac{9}{4} \int e^{2x} \sin(3x) \;\; dx$$

$$\frac{13}{4} \int e^{2x} \sin 3x \; dx = \frac{1}{2} e^{2x} \sin(3x) - \frac{3}{4} e^{2x} \cos(3x)$$

$$\int e^{2x} \sin 3x \; dx = \frac{4}{13} \left(\frac{1}{2} e^{2x} \sin(3x) - \frac{3}{4} e^{2x} \cos(3x) \right) + C$$

$$\int e^{2x} \sin 3x \; dx = \frac{4}{13} \left(\frac{2e^{2x} \sin(3x) - 3e^{2x} \cos(3x)}{4} \right) + C$$

$$\int e^{2x} \sin 3x \; dx = \frac{2e^{2x} \sin(3x) - 3e^{2x} \cos(3x)}{13} + C$$

5. (5 points) Integrate the following:

$$\int x^4 e^{2x}\, dx$$

u		dv
x^4	$+$	e^{2x}
$4x^3$	$-$	$\frac{e^{2x}}{2}$
$12x^2$	$+$	$\frac{e^{2x}}{4}$
$24x$	$-$	$\frac{e^{2x}}{8}$
24	$+$	$\frac{e^{2x}}{16}$
0		$\frac{e^{2x}}{32}$

$$\int x^4 e^{2x}\, dx = \frac{1}{2}x^4 e^{2x} - \frac{4}{4}x^3 e^{2x} + \frac{12}{8}x^2 e^{2x} - \frac{24}{16}xe^{2x} + \frac{24}{32}e^{2x} + C$$

$$= \frac{1}{2}x^4 e^{2x} - x^3 e^{2x} + \frac{3}{2}x^2 e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{4}e^{2x} + C$$

$$= \frac{e^{2x}}{4}\left(2x^4 - 4x^3 + 6x^2 - 6x + 3\right) + C$$

6. (5 points) Integrate the following:

$$\int \sec^3 x \, \tan^5 x \, dx$$

Note that $\sin^2 x + \cos^2 x = 1$ *so that, dividing by* $\cos^2 x$, *we have* $\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$ *which implies* $\tan^2 x + 1 = \sec^2 x$. *But then* $\tan^2 x = \sec^2 x - 1$ *and* $\tan^4 x = (\tan^2 x)^2 = (\sec^2 x - 1)^2$. *Now let* $u = \sec x$. *Then* $du = \sec x \tan x \, dx$. *Note that* $\tan^4 x = (\sec^2 x - 1)^2 = (u^2 - 1)^2$. *Now*

$$\int \sec^3 x \, \tan^5 x \, dx = \int \sec^2 x \, \tan^4 x \cdot \sec x \tan x \, dx$$

$$= \int u^2 (u^2 - 1)^2 \, du$$

$$= \int u^2 (u^4 - 2u^2 + 1) \, du$$

$$= \int u^6 - 2u^4 + u^2 \, du$$

$$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C$$

$$= \frac{\sec^7 x}{7} - \frac{2 \sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

$$= \frac{15 \sec^7 x - 41 \sec^5 x + 35 \sec^3 x}{105} + C$$

7. (5 points) Integrate the following:

$$\int \frac{dx}{x^{2017} - x}$$

$$\int \frac{dx}{x^{2017} - x} = \int \frac{dx}{x(x^{2016} - 1)} = \int \frac{1}{x(x^{2016} - 1)} \cdot \frac{x^{2015}}{x^{2015}}\, dx = \int \frac{x^{2015}}{x^{2016}(x^{2016} - 1)}\, dx$$

Let $u = x^{2016}$*. Then* $du = 2016x^{2015}\, dx$ *so that* $dx = \dfrac{du}{2016x^{2015}}$*. Then*

$$\int \frac{x^{2015}}{x^{2016}(x^{2016} - 1)}\, dx = \frac{1}{2016} \int \frac{du}{u(u-1)}$$

But we have $\dfrac{1}{u(u-1)} = \dfrac{A}{u} + \dfrac{B}{u-1}$*. Using Heaviside's Method, we have* $A = \dfrac{1}{0-1} = -1$ *and* $B = \dfrac{1}{1} = 1$*. Then*

$$\begin{aligned}\int \frac{dx}{x^{2017} - x} &= \frac{1}{2016} \int \frac{du}{u(u-1)} = \frac{1}{2016} \int \frac{1}{u-1} - \frac{1}{u}\, du \\ &= \frac{1}{2016} \left(\ln|u-1| - \ln|u| \right) + C = \frac{1}{2016} \ln\left| \frac{u-1}{u} \right| + C = \frac{1}{2016} \ln\left| \frac{x^{2016} - 1}{x^{2016}} \right| + C\end{aligned}$$

<u>OR</u>

$$\int \frac{dx}{x^{2017} - x} = \int \frac{dx}{x^{2017}\left(1 - \frac{1}{x^{2016}}\right)}$$

Now let $u = 1 - \dfrac{1}{x^{2016}}$ *so that* $du = \dfrac{2016}{x^{2017}}\, dx$ *so that* $dx = \dfrac{x^{2017}}{2016}\, du$*. Then*

$$\begin{aligned}\int \frac{dx}{x^{2017} - x} &= \int \frac{dx}{x^{2017}\left(1 - \frac{1}{x^{2016}}\right)} = \int \frac{x^{2017}}{x^{2017}u} \frac{du}{2016} = \frac{1}{2016} \int \frac{du}{u} \\ &= \frac{1}{2016} \ln|u| + C = \frac{1}{2016} \ln\left| 1 - \frac{1}{x^{2016}} \right| + C = \frac{1}{2016} \ln\left| \frac{x^{2016}}{x^{2016}} - \frac{1}{x^{2016}} \right| + C \\ &= \frac{1}{2016} \ln\left| \frac{x^{2016} - 1}{x^{2016}} \right| + C\end{aligned}$$

8. (5 points) Integrate the following:

$$\int \frac{3x-1}{x^3+x^2+x+1}\,dx$$

$$\frac{3x-1}{x^3+x^2+x+1} = \frac{3x-1}{x^2(x+1)+(x+1)} = \frac{3x-1}{(x+1)(x^2+1)}$$

$$\frac{3x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

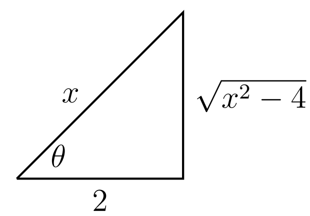
Using Heaviside's Method, we find $A = \frac{3(-1)-1}{1+1} = -2$. *Now if* $x = 0$, *we have* $\frac{3(0)-1}{1(1)} = \frac{-2}{1} + \frac{0+C}{0+1}$ *so that* $-1 = -2 + C$, *which in turn implies* $C = 1$. *Now if* $x = 1$, *we have* $\frac{3-1}{2(2)} = \frac{-2}{2} + \frac{B+1}{2}$ *so that* $\frac{1}{2} = -1\frac{B+1}{2}$ *which in turn implies that* $1 = -2 + B + 1$ *so that* $B = 2$. *Then*

$$\begin{aligned}
\int \frac{3x-1}{x^3+x^2+x+1}\,dx &= \int \frac{-2}{x+1} + \frac{2x+1}{x^2+1}\,dx \\
&= \int \frac{-2}{x+1} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\,dx \\
&= -2\ln|x+1| + \ln|x^2+1| + \arctan x + K \\
&= \arctan x + \ln\left|\frac{x^2+1}{(x+1)^2}\right| + K
\end{aligned}$$

9. (5 points) Integrate the following:

$$\int \frac{\sqrt{x^2-4}}{x}\,dx$$

$$a^2 + b^2 = c^2$$
$$b^2 = \underbrace{c^2 - a^2}_{x^2 - 2^2}$$



$$\cos\theta = \frac{2}{x}$$
$$x = 2\sec\theta$$
$$dx = 2\sec\theta\tan\theta\,d\theta$$

$$\tan\theta = \frac{\sqrt{x^2-4}}{2}$$
$$\sqrt{x^2-4} = 2\tan\theta$$

$$\begin{aligned}\int \frac{\sqrt{x^2-4}}{x}\,dx &= \int \frac{2\tan\theta}{2\sec\theta}\cdot 2\sec\theta\tan\theta\,d\theta \\ &= 2\int \tan^2\theta\,d\theta \\ &= 2\int (\sec^2\theta - 1)\,d\theta \\ &= 2(\tan\theta - \theta) + C\end{aligned}$$

Now $x = 2\sec\theta$ *so that* $\theta = \sec^{-1}(x/2)$ *and* $\tan\theta = \dfrac{\sqrt{x^2-4}}{2}$. *Then*

$$\int \frac{\sqrt{x^2-4}}{x}\,dx = 2(\tan\theta - \theta) + C = 2\left(\frac{\sqrt{x^2-4}}{2} - \sec^{-1}(x/2)\right) + C = \sqrt{x^2-4} - 2\sec^{-1}\left(\frac{x}{2}\right) + C$$