Math 296: Final Exam
Summer Session II – 2016
08/11/2016
145 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 25 pages (including this cover page) and 24 questions. Check that you have every page of the exam.

Do seven questions from Problems 1 to 9, six questions from Problems 10 to 17, and five questions from Problems 18 to 24 – for a total of eighteen problems. Clearly indicate which problems you wish to be graded by placing a checkmark in the circle at the top of the page. Only questions which have a checkmark will be graded.

For each question, write your answer in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page – being sure to indicate the problem number. Each question is out of 5 points.

Problem	Score	Problem	Score	Problem	Score
1		10		18	
2		11		19	
3		12		20	
4		13		21	
5		14		22	
6		15		23	
7		16		24	
8		17			
9					
Total		Total		Total	

Exam Total	Average	Percentage

1. (5 points) Find the limits, if they exist, of the following sequences:

$$\lim_{n\to\infty} \cos(1/n) + \sqrt[n]{2n} = \cos(0) + 1 = 2$$

$$\lim_{n\to\infty} \frac{2n^2 - 3n + 7}{3n^2 + 2n + 5} = 2/3$$

$$\lim_{n\to\infty} \ln(2n+3) - \ln(3n-5) = \lim_{n\to\infty} \ln\left(\frac{2n+3}{3n-1}\right) = \ln\left(\lim_{n\to\infty} \frac{2n+3}{3n+5}\right) = \ln(2/3)$$

$$\lim_{n\to\infty} \sin\left(\frac{n\pi}{2}\right) = \textit{DNE}$$

$$\lim_{n\to\infty} \left(1 + \frac{2}{3n}\right)^{3n/5}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{3n/2}\right)^{3n/5} = \lim_{n\to\infty} \left(1 + \frac{1}{3n/2}\right)^{3n/5 \cdot 2/2} = \lim_{n\to\infty} \left[\left(1 + \frac{1}{3n/2}\right)^{3n/2}\right]^{2/5} = e^{2/5} = \sqrt[5]{e^2}$$

2. (5 points) Determine whether the following series converge or diverge. Be sure to justify your answer.

$$\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} n\sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 \neq 0$$

Therefore, $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$ *diverges by the Divergence Test.*

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$$

Note that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the* p*-test (in fact, it converges to* $\pi^2/6$*). Now*

$$\lim_{n\to\infty} \frac{\sin(1/n^2)}{1/n^2} = 1 > 0$$

Because $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ *converges if and only if* $\sum_{n=1}^{\infty} 1/n^2$ *converges by the Limit Comparison Test,* $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ *converges.*

3. (5 points) Determine if the following series converge or diverge. If the series converges, find the sum. If the series diverges, explain why.

$$\sum_{n=1}^{\infty} \frac{5\pi^{n+1}}{3^{n-1}}$$

$$\sum_{n=1}^{\infty} \frac{5\pi^{n+1}}{3^{n-1}} = \sum_{n=1}^{\infty} 5\left(\frac{\pi}{3}\right)^n$$

This series is geometric with $r = \pi/3$. But $\pi \approx 3.14159\ldots$ so that $|r| = |\pi/3| > 1$. Therefore, this series diverges by the Geometric Series Test. Note that $\lim_{n\to\infty} \frac{5\pi^n}{3^n} = \infty \neq 0$ so that the divergence is immediate from the Divergence Theorem.

$$\sum_{n=1}^{\infty} \frac{3^n + 1}{2^{2n}}$$

$$\sum_{n=1}^{\infty} \frac{3^n + 1}{2^{2n}} = \sum_{n=1}^{\infty} \left(\frac{3^n}{2^{2n}} + \frac{1}{2^{2n}}\right) = \sum_{n=1}^{\infty} \left(\frac{3^n}{4^n} + \frac{1}{4^n}\right) = \sum_{n=1}^{\infty} \left[\left(\frac{3}{4}\right)^n + \left(\frac{1}{4}\right)^n\right]$$

Now note that $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ is geometric with $r = 3/4$ and $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$ is geometric with $r = 1/4$. Because in each case $|r| < 1$, both these series converge by the Geometric Series Test. Furthermore,

$$\sum_{n=1}^{\infty} \left(\frac{3^n}{4^n} + \frac{1}{4^n}\right) = \frac{3/4}{1 - 3/4} + \frac{1/4}{1 - 1/4} = \frac{3/4}{1/4} + \frac{1/4}{3/4} = 3 + \frac{1}{3} = \frac{10}{3}$$

4. (5 points) Determine if the following series converges or diverges. If the series converges, find the sum. If the series diverges, explain why.

$$\sum_{n=0}^{\infty} \frac{2}{n^2+4n+3}$$

$$\sum_{n=0}^{\infty} \frac{2}{n^2+4n+3} = 2\sum_{n=0}^{\infty} \frac{1}{n^2+4n+3} = 2\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+3)}$$

But we can apply a partial fraction decomposition: $\dfrac{1}{(n+1)(n+3)} = \dfrac{A}{n+1} + \dfrac{B}{n+3}$. *Using Heaviside's Method, we find* $A = \dfrac{1}{-1+3} = \dfrac{1}{2}$ *and* $B = \dfrac{1}{-3+1} = \dfrac{-1}{2}$. *Then*

$$2\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+3)} = 2\sum_{n=0}^{\infty} \frac{1/2}{n+1} + \frac{-1/2}{n+3} = \sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$$

But then we have

$$\begin{aligned}\sum_{n=0}^{M} \frac{1}{n+1} - \frac{1}{n+3} &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \\ &\quad \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{M} - \frac{1}{M+2}\right) + \left(\frac{1}{M+1} - \frac{1}{M+3}\right) \\ &= 1 + \frac{1}{2} - \frac{1}{M+2} - \frac{1}{M+3}\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{2}{n^2+4n+3} = \lim_{M\to\infty} \sum_{n=0}^{M} \frac{1}{n+1} - \frac{1}{n+3} = \lim_{M\to\infty} 1 + \frac{1}{2} - \frac{1}{M+2} - \frac{1}{M+3} = 1 + \frac{1}{2} + 0 + 0 = \frac{3}{2}$$

Intuitively, this is because

$$\sum_{n=0}^{\infty} \frac{2}{n^2+4n+3} = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots$$

5. (5 points) Determine if the following series converge or diverge. Be sure to justify your answers.

$$\sum_{n=1}^{\infty} \frac{2n-5}{\sqrt{3n^6-2}}$$

The convergence/divergence of a series does not depend on the first few terms; therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{2n-5}{\sqrt{3n^6-2}}$ *converges if and only if* $\displaystyle\sum_{n=3}^{\infty} \frac{2n-5}{\sqrt{3n^6-2}}$ *converges. [We care about the second series because all the terms are positive.] Now*

$$0 \leq \sum_{n=3}^{\infty} \frac{2n-5}{\sqrt{3n^6-2}} \leq \sum_{n=1}^{\infty} \frac{2n}{\sqrt{3n^6-2n^6}} = \sum_{n=1}^{\infty} \frac{2n}{n^3} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}$$

where $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the* p*-test. Therefore,* $\displaystyle\sum_{n=3}^{\infty} \frac{2n-5}{\sqrt{3n^6-2}}$ *converges by the Comparison Test so that the series* $\displaystyle\sum_{n=1}^{\infty} \frac{2n-5}{\sqrt{3n^6-2}}$ *converges.*

$$\sum_{n=1}^{\infty} \frac{7+4\sqrt[3]{n^4}}{n^2+2n+1}$$

$$\sum_{n=1}^{\infty} \frac{7+4\sqrt[3]{n^4}}{n^2+2n+1} \geq \sum_{n=1}^{\infty} \frac{4\sqrt[3]{n^4}}{n^2+2n^2+n^2} = \sum_{n=1}^{\infty} \frac{4\sqrt[3]{n^4}}{4n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}} \geq 0$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ *diverges by the* p*-test; therefore,* $\displaystyle\sum_{n=1}^{\infty} \frac{7+4\sqrt[3]{n^4}}{n^2+2n+1}$ *diverges by the Comparison Test.*

6. (5 points) Determine if the following series converges or diverges. If the series converges, determine at most how many terms are needed to approximate the sum to 4 decimal digits of accuracy. If the series diverges, explain why.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$$

Note that this is an alternating series and

(i) $\lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$

(ii) $\left\{\frac{1}{1+\sqrt{n}}\right\}$ *is a decreasing sequence. [Note that* $f(x) := \sqrt{x}$ *is an increasing function because* $f'(x) = \frac{1}{2\sqrt{x}} > 0$*. Then* $\sqrt{n+1} > \sqrt{n}$ *so that* $1+\sqrt{n+1} > 1+\sqrt{n}$ *which implies* $\frac{1}{1+\sqrt{n+1}} < \frac{1}{1+\sqrt{n}}$*, proving the sequence is decreasing.]*

Therefore by the Alternating Series Test, the series converges. To sum the series to 4 decimal digits, we want to sum until the terms of the sequence are at most $1/10^4$*. Now if we choose* n *such that* $1/\sqrt{n} < 1/10^4$*, then* $\sqrt{n} > 10^4$*. But then* $n > 10^8 = 100,000,000$*. So if* $n = 100,000,001$*, then*

$$\frac{1}{10^4} > \frac{1}{\sqrt{n}} > \frac{1}{1+\sqrt{n}}$$

so that at most $n = 100,000,001$ *terms are needed.*

7. (5 points) Determine if the following series converge or diverge. Be sure to justify your answer.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$\lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \cancel{(n+1)} \cdot \frac{n^n}{(n+1)^n \cancel{(n+1)}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^{-n} = \lim_{n\to\infty} (1+1/n)^{-n} = \lim_{n\to\infty} \left[(1+1/n)^n \right]^{-1} = e^{-1} = \frac{1}{e} < 1$$

Therefore, $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ *converges absolutely by the Ratio Test.*

$$\sum_{n=1}^{\infty} \left(\frac{2n^3+5n^2-n+6}{3n^3+6n^2-n+7} \right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n^3+5n^2-n+6}{3n^3+6n^2-n+7} \right)^n} = \lim_{n\to\infty} \frac{2n^3+5n^2-n+6}{3n^3+6n^2-n+7} = \frac{2}{3} < 1$$

Therefore, $\sum_{n=1}^{\infty} \left(\frac{2n^3+5n^2-n+6}{3n^3+6n^2-n+7} \right)^n$ *converges absolutely by the Root Test.*

8. (5 points) Determine the interval and radius of convergence of the following power series:

$$\sum_{n=1}^{\infty}(-1)^n \frac{(x+2)^n}{n\,2^n}$$

Using the Root Test, we find

$$\lim_{n\to\infty} \sqrt[n]{(-1)^n \frac{(x+2)^n}{n\,2^n}} = \lim_{n\to\infty} \frac{|x+2|}{\sqrt[n]{n}\cdot 2} = \frac{|x+2|}{2}$$

We want $\frac{|x+2|}{2} < 1$ *so that* $-1 < \frac{|x+2|}{2} < 1$. *But then* $-2 < x+2 < 2$ *so that* $-4 < x < 0$. *Using the Ratio Test, we find*

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(-1)^n(x+2)^n}\right| &= \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(x+2)^n}\cdot\frac{n}{n+1}\cdot\frac{2^n}{2^{n+1}}\right| \\ &= \lim_{n\to\infty}\left|(x+2)\cdot\frac{n}{n+1}\cdot\frac{1}{2}\right| = \frac{|x+2|}{2}\end{aligned}$$

We want $\frac{|x+2|}{2} < 1$ *so that* $-1 < \frac{|x+2|}{2} < 1$. *But then* $-2 < x+2 < 2$ *so that* $-4 < x < 0$.

Now if $x = 0$, *the series is* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. *But* $\lim_{n\to\infty} \frac{1}{n} = 0$ *and the sequence* $\left\{\frac{1}{n}\right\}$ *is decreasing. Therefore,* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *converges by the Alternating Series Test.*

Now if $x = -4$, *the series is* $\sum_{n=1}^{\infty} \frac{1}{n}$. *This series diverges by the* p*-test. But then we have*

$$\text{Interval of Convergence} = (-4, 0]$$

$$R = \frac{0-(-4)}{2} = 2$$

9. (5 points) Use known Taylor/Maclaurin Series to evaluate the following:

$$\int x \cos x^3 \, dx$$

Using the Taylor Series for $\cos x$*:*

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)x^{2n}}{(2n)!} \\
\cos x^3 &= \sum_{n=0}^{\infty} \frac{(-1)(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)x^{6n}}{(2n)!} \\
x \cos x^3 &= \sum_{n=0}^{\infty} \frac{(-1)x^{6n+1}}{(2n)!}
\end{aligned}$$

But then

$$\int x \cos x^3 \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)x^{6n+1}}{(2n)!} \, dx = \sum_{n=0}^{\infty} \int \frac{(-1)x^{6n+1}}{(2n)!} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(6n+2)(2n)!} + C$$

$$\sum_{n=0}^{\infty} \frac{n}{5^n}$$

We us the Taylor series for $\frac{1}{1-x}$*:* $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ *for* $|x| < 1$*. Then*

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx} \, x^n = \sum_{n=0}^{\infty} n x^{n-1} \\
x \, \frac{d}{dx}\left(\frac{1}{1-x}\right) &= x \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} n x^n \\
x \, \frac{d}{dx}\left(\frac{1}{1-x}\right) &= x \cdot \frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}
\end{aligned}$$

Now the given series is the series above with $x = 1/5$*. Then*

$$\sum_{n=0}^{\infty} \frac{n}{5^n} = \sum_{n=0}^{\infty} n \left(\frac{1}{5}\right)^n = \frac{x}{(1-x)^2}\bigg|_{x=1/5} = \frac{1/5}{(1-1/5)^2} = \frac{1/5}{16/25} = \frac{5}{16}$$

Note there is an elementary way of evaluating the sum! Can you find it?

10. (5 points) Integrate the following:

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$$

Let $u = e^x + e^{-x}$*. Then* $du = (e^x - e^{-x})\, dx$ *so that* $dx = \frac{du}{e^x - e^{-x}}$*. But then*

$$\int \frac{du}{u} = \ln|u| + C = \ln|e^x + e^{-x}| + C$$

Alternatively, $\frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x$*. Then*

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx = \int \tanh x\, dx = \ln|\cosh x| + C$$

$$= \ln\left|\frac{e^x + e^{-x}}{2}\right| + C = \ln|e^x + e^{-x}| - \ln 2 + C = \ln|e^x + e^{-x}| + C$$

$$\int \frac{e^{2x}}{e^{2x} + 1}\, dx$$

Let $u = e^{2x} + 1$*. Then* $du = 2e^{2x}\, dx$ *so that* $dx = \frac{du}{2e^{2x}}$*. Then*

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|e^{2x} + 1| + C = \ln\sqrt{e^{2x} + 1} + C$$

11. (5 points) A table of integrals gives the following:

$$\int \frac{dx}{x^3(x^4-a^4)} = \frac{1}{2a^4x^2} + \frac{1}{4a^6}\ln\left(\frac{x^2-a^2}{x^2+a^2}\right)$$

Use this to evaluate the following:

$$\int \frac{e^{-2x}}{e^{4x}-1}\,dx$$

$$\int \frac{e^{-2x}}{e^{4x}-1}\,dx = \int \frac{1}{e^{2x}(e^{4x}-1)}\,dx = \int \frac{1}{e^{2x}(e^{4x}-1)}\cdot\frac{e^x}{e^x}\,dx = \int \frac{e^x}{e^{3x}(e^{4x}-1)}\,dx$$

Now let $u = e^x$. Then $du = e^x\,dx$ so that $dx = \dfrac{du}{e^x}$. But then using the formula above with $a = 1$,

$$\int \frac{e^x}{e^{3x}(e^{4x}-1)}\,dx = \int \frac{du}{u^3(u^4-1)} = \frac{1}{2u^2}+\frac{1}{4}\ln\left(\frac{u^2-1}{u^2+1}\right)+C = \frac{1}{2e^{2x}}+\frac{1}{4}\ln\left(\frac{e^{2x}-1}{e^{2x}+1}\right)+C$$

12. (5 points) Evaluate the following integrals:

$$\int x^4 \ln x \, dx$$

$\ln x$	$\frac{x^5}{5}$
$\frac{1}{x}$	x^4

$$\frac{1}{5}x^5 \ln x - \int \frac{x^4}{5} \, dx = \frac{x^5 \ln x}{5} - \frac{x^5}{25} + C = \frac{5x^5 \ln x - x^5}{25} + C = \frac{x^5(5 \ln x - 1)}{25} + C = \frac{x^5(\ln x^5 - 1)}{25} + C$$

$$\int \frac{x^5}{\sqrt{1-x^3}} \, dx$$

x^3	$\frac{-2}{3}\sqrt{1-x^3}$
$3x^2$	$\frac{x^2}{\sqrt{1-x^3}}$

$$\frac{-2}{3}x^3\sqrt{1-x^2} + \int 2x^2\sqrt{1-x^3} \, dx = \frac{-2}{3}x^3\sqrt{1-x^3} - \frac{4}{9}\sqrt{(1-x^3)^3} + C = \frac{-2}{9}\sqrt{1-x^3}(x^3+2) + C$$

13. (5 points) Evaluate the following integral:

$$\int x^4 \sin(2x)\, dx$$

u	sign	dv
x^4	$+$	$\sin(2x)$
$4x^3$	$-$	$-\frac{\cos(2x)}{2}$
$12x^2$	$+$	$-\frac{\sin(2x)}{4}$
$24x$	$-$	$\frac{\cos(2x)}{8}$
24	$+$	$\frac{\sin(2x)}{16}$
0		$\frac{\sin(2x)}{16}$

$$\frac{-1}{2}x^4\cos(2x) + \frac{4}{4}x^3\sin(2x) + \frac{12}{8}x^2\cos(2x) - \frac{24}{16}x\sin(2x) - \frac{24}{32}\cos(2x) + C$$

$$\frac{-1}{2}x^4\cos(2x) + x^3\sin(2x) + \frac{3}{2}x^2\cos(2x) - \frac{3}{2}x\sin(2x) - \frac{3}{4}\cos(2x) + C$$

14. (5 points) Evaluate the following integral:

$$\int e^x \sin 2x \; dx$$

u	sign	dv
$\sin(2x)$	$+$	e^x
$2\cos(2x)$	$-$	e^x
$-4\sin(2x)$	$+$	e^x

$$\int e^x \sin 2x \; dx = e^x \sin(2x) - 2e^x \cos(2x) - \int 4e^x \sin(2x) \; dx$$

$$\int e^x \sin 2x \; dx = e^x \sin(2x) - 2e^x \cos(2x) - 4\int e^x \sin(2x) \; dx$$

$$5\int e^x \sin 2x \; dx = e^x \sin(2x) - 2e^x \cos(2x)$$

$$\int e^x \sin 2x \; dx = \frac{e^x \sin(2x) - 2e^x \cos(2x)}{5} + C$$

$$\int e^x \sin 2x \; dx = \frac{e^x\big(\sin(2x) - 2\cos(2x)\big)}{5} + C$$

15. (5 points) Evaluate the following integral:

$$\int \frac{3x^2 + x - 2}{x^3 - x^2 + x - 1}\, dx$$

We have

$$\frac{3x^2 + x - 2}{x^3 - x^2 + x - 1} = \frac{3x^2 + x - 2}{x^2(x-1) + (x-1)} = \frac{3x^2 + x - 2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1}$$

Using Heaviside's, we find $A = \frac{3+1-2}{1+1} = \frac{2}{2} = 1$*. Now if* $x = 0$*, we have* $\frac{0+0-2}{-1\cdot 1} = \frac{1}{-1} + \frac{0+C}{1}$ *so that* $2 = -1 + C$*, which implies* $C = 3$*. Finally, if* $x = -1$ *then* $\frac{3-1-2}{-2\cdot 2} = \frac{1}{-2} + \frac{-B+3}{2}$ *so that* $0 = \frac{-1}{2} + \frac{-B+3}{2}$ *which implies* $-B + 3 - 1 = 0$*, meaning* $B = 2$*. Then*

$$\begin{aligned}
\int \frac{3x^2 + x - 2}{x^3 - x^2 + x - 1}\, dx &= \int \frac{1}{x-1} + \frac{2x+3}{x^2+1}\, dx \\
&= \int \frac{1}{x-1} + \frac{2x}{x^2+1} + \frac{3}{x^2+1}\, dx \\
&= \ln|x-1| + \ln|x^2+1| + 3\arctan x + K \\
&= \ln|(x-1)(x^2+1)| + 3\arctan x + K
\end{aligned}$$

16. (5 points) Evaluate the following integral:

$$\int \sin^2 x \, \cos^3 x \; dx$$

Note that $\sin^2 x + \cos^2 x = 1$ *so that* $\cos^2 x = 1 - \sin^2 x$. *Now*

$$\int \sin^2 x \, \cos^3 x \; dx = \int \sin^2 x \, \cos^2 x \cos x \; dx = \int \sin^2 (1 - \sin^2 x) \cos x \; dx$$

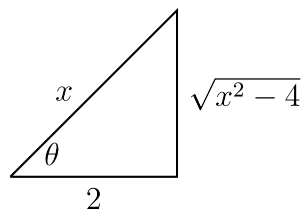
Let $u = \sin x$. *Then* $du = \cos x \; dx$. *Then we have*

$$\int u^2(1-u^2) \; du = \int (u^2-u^4) \; du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{5u^3 - 3u^5}{15} + C = \frac{5\sin^3 x - 3\sin^5 x}{15} + C$$

17. (5 points) Integrate the following:

$$\int \frac{\sqrt{x^2-4}}{x^3}\,dx$$

$$a^2+b^2=c^2$$
$$b^2=\underbrace{c^2-a^2}_{x^2-4}$$



$$\cos\theta=\frac{2}{x}$$
$$x=\frac{2}{\cos\theta}$$
$$x=2\sec\theta$$
$$dx=2\sec\theta\tan\theta\;d\theta$$

$$\tan\theta=\frac{\sqrt{x^2-4}}{2}$$
$$\sqrt{x^2-4}=2\tan\theta$$

$$\begin{aligned}
\int \frac{\sqrt{x^2-4}}{x^3}\,dx &= \int \frac{2\tan\theta}{8\sec^3\theta}2\sec\theta\tan\theta\;d\theta = \frac{1}{2}\int\frac{\tan^2\theta}{\sec^2\theta}\;d\theta\\
&= \frac{1}{2}\int \tan^2\theta\cos^2\theta\;d\theta = \frac{1}{2}\int \sin^2\theta\;d\theta\\
&= \frac{1}{2}\int \frac{1-\cos 2\theta}{2}\;d\theta = \frac{1}{4}\int (1-\cos 2\theta)\;d\theta\\
&= \frac{1}{4}\left(\theta-\frac{\sin 2\theta}{2}\right)+C = \frac{\theta}{4}-\frac{\sin 2\theta}{8}+C\\
&= \frac{\theta}{4}-\frac{2\sin\theta\cos\theta}{8}+C = \frac{\theta}{4}-\frac{\sin\theta\cos\theta}{4}+C = \frac{1}{4}(\theta-\sin\theta\cos\theta)+C
\end{aligned}$$

But $\sin\theta=\dfrac{\sqrt{x^2-4}}{x}$ ***and*** $\cos\theta=\dfrac{2}{x}$. ***But also*** $x=2\sec\theta$ ***so that*** $\sec\theta=\dfrac{x}{2}$, ***which implies*** $\theta=\sec^{-1}\left(\dfrac{x}{2}\right)$. ***Then***

$$\begin{aligned}
\int \frac{\sqrt{x^2-4}}{x^3}\,dx &= \frac{1}{4}(\theta-\sin\theta\cos\theta)+C\\
&= \frac{1}{4}\left(\sec^{-1}\left(\frac{x}{2}\right)-\frac{\sqrt{x^2-4}}{x}\cdot\frac{2}{x}\right)+C\\
&= \frac{1}{4}\left(\sec^{-1}\left(\frac{x}{2}\right)-\frac{2\sqrt{x^2-4}}{x^2}\right)+C
\end{aligned}$$

18. (5 points) Evaluate the following integral:

$$\int_1^\infty \frac{dx}{2x^2+x}$$

$$\frac{1}{2x^2+x} = \frac{1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$$

Using Heaviside's Method, we have $A = \dfrac{1}{2(0)+1} = \dfrac{1}{1} = 1$ *and* $B = \dfrac{1}{-1/2} = -2$. *Then*

$$\int_1^\infty \frac{dx}{2x^2+x} = \lim_{b\to\infty} \int_1^b \frac{dx}{2x^2+x}$$

$$= \lim_{b\to\infty} \int_1^b \left(\frac{1}{x} - \frac{2}{2x+1}\right)\, dx$$

$$= \lim_{b\to\infty} \big(\ln|x| - \ln|2x+1|\big)\Big|_1^b$$

$$= \lim_{b\to\infty} \ln\left|\frac{x}{2x+1}\right| \Big|_1^b$$

$$= \lim_{b\to\infty} \ln\left|\frac{b}{2b+1}\right| - \ln(1/3)$$

$$= \ln(1/2) - \ln(1/3)$$

$$= -\ln 2 + \ln 3$$

$$= \ln(3/2)$$

19. (5 points) Complete the following parts:

(a) Convert the polar coordinate $(2, 5\pi/3)$ to cartesian coordinates.

We have $r = 2$ *and* $\theta = \dfrac{5\pi}{3}$*. Then*

$$\begin{aligned} x &= r\cos\theta = 2\cos\frac{5\pi}{3} = 1 \\ y &= r\sin\theta = 2\sin\frac{5\pi}{3} = -\sqrt{3} \end{aligned}$$

Then $(2, 5\pi/3) \sim (1, -\sqrt{3})$.

(b) Convert the cartesian coordinate $(-1, \sqrt{3})$ to polar coordinates.

We have $r^2 = x^2 + y^2 = (-1)^2 + (\sqrt{3})^2 = 1 + 3 = 4$ *so that* $r = 2$*. We also have* $\theta = \tan^{-1}(y/x) = \tan^{-1}(\sqrt{3}/-1) = \dfrac{2\pi}{3}$*. Then* $(-1, \sqrt{3}) \sim (2, 2\pi/3)$.

(c) Find the equation of the tangent line of $r(\theta) = 1 + \sin\theta$ at $\theta = 0$.

We have $r(\theta) = 1 + \sin\theta$*. Then* $r'(\theta) = \cos\theta$*. We know that* $y = r\sin\theta$ *so that* $y' = r'\sin\theta + r\cos\theta$*. We also know* $x = r\cos\theta$ *so that* $x' = r'\cos\theta - r\sin\theta$*. When* $\theta = 0$*, we have* $r(0) = 1 + \sin 0 = 1$*. But then using* $\theta = 0$ *and* $r = 1$*, we find* $x = 1$ *and* $y = 0$*. Now*

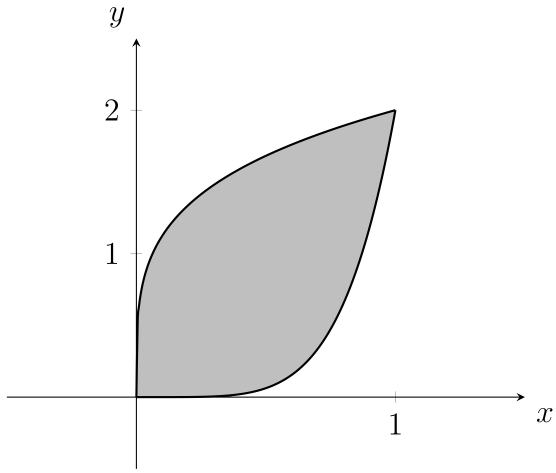
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{y'}{x'} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

Evaluating at $\theta = 0$*, we find* $\dfrac{dy}{dx} = \dfrac{1(0) + (1+0)1}{1(1) - (1+0)0} = \dfrac{1}{1} = 1$*. Therefore,*

$$\begin{aligned} y - 0 &= 1(x - 1) \\ y &= x - 1 \end{aligned}$$

20. (5 points) Set up the integrals to calculate—but do not evaluate—the following:

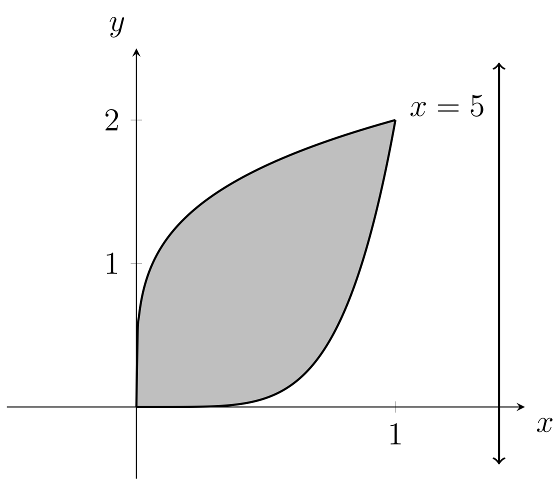
(a) The area between the curves $y = 2\sqrt[4]{x}$ and $y = 2x^5$.



Now $y = 2\sqrt[4]{x} \Leftrightarrow x = \dfrac{y^4}{16}$, and $y = 2x^5 \Leftrightarrow x = \sqrt[5]{\dfrac{y}{2}}$. Then the area is

$$\int_0^1 2\sqrt[4]{x} - 2x^5\, dx \quad \textbf{\underline{OR}} \quad \int_0^2 \frac{y^4}{16} - \sqrt[5]{\frac{y}{2}}\, dy$$

(b) The volume revolving the region bound by the curves $y = 2\sqrt[4]{x}$ and $y = 2x^5$ about the line $x = 5$. You must set-up the integrals using both the Disk/Washer Method and the Shell Method.



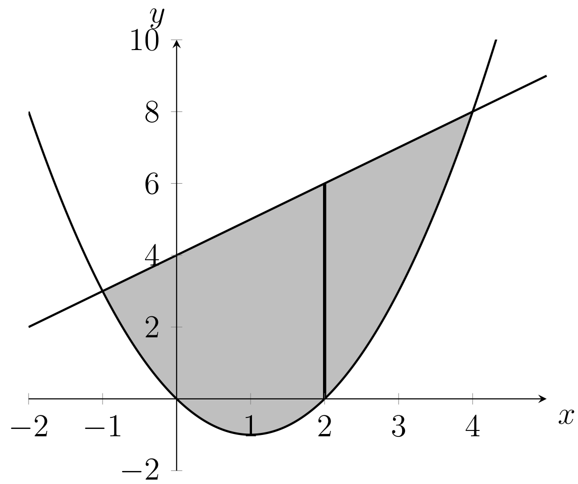
Now $y = 2\sqrt[4]{x} \Leftrightarrow x = \dfrac{y^4}{16}$, *and* $y = 2x^5 \Leftrightarrow x = \sqrt[5]{\dfrac{y}{2}}$. *Then the area is*

$$\pi \int_0^2 \left(5 - \frac{y^4}{16}\right)^2 - \left(5 - \sqrt[5]{\frac{y}{2}}\right)^2\, dy$$

<u>OR</u>

$$2\pi \int_0^1 (5 - x)(2\sqrt[4]{x} - 2x^5)\, dx$$

21. (5 points) The base of a solid is given by the region bound by $y = x^2 - 2x$ and $y = x+4$. Cross sections of this solid perpendicular to the x-axis are isosceles right triangles with hypotenuse lying in the plane. Set up the integral to calculate—but do not evaluate—the volume of the solid.



Now $x^2 - 2x = x + 4$ *imlpies* $x^2 - 3x - 4 = 0$. *But* $x^2 - 3x - 4 = (x+1)(x-4)$ *so that either* $x = -1$ *or* $x = 4$. *Given an isosceles triangle with legs of length* s *and hypotenuse* b, *using the Pythagorean Theorem, we have* $s^2 + s^2 = b^2$. *Then* $2s^2 = b^2$ *so that* $\sqrt{2}s = b$. *Therefore,* $s = \dfrac{b}{\sqrt{2}}$. *But the hypotenuse lies along the plane, so it must have length equal to the distance between the curves at a given* x*-value, e.g. the black segment in the diagram. This gives* $b = (x+4) - (x^2 - 2x) = x + 4 - x^2 + 3x = 4x - x^2 + 4$. *The area of a triangle is* $A = \dfrac{1}{2}s \cdot s = \dfrac{s^2}{2} = \dfrac{b^2/2}{2} = \dfrac{b^2}{4}$. *Therefore,*

$$V = \int_{-1}^{4} A(x)\, dx = \int_{-1}^{4} \frac{1}{4}\left(4x - x^2 + 4\right)^2\, dx$$

22. (5 points) Set up the integrals to calculate—but do not evaluate—the following:

(a) The length of the curve $y^2 - 4x = 0$ from $(1, 2)$ to the point $(4, 4)$.

Note that if $y^2 - 4x = 0$ so that $x = \frac{y^2}{4}$. Then $x' = \frac{y}{2}$. Equivalently, $y = 2\sqrt{x}$ (the positive root because our y's are positive) so that $y' = \frac{1}{\sqrt{x}}$. Then the length is

$$\int_1^4 \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{1}{x}}\, dx$$

OR

$$\int_2^4 \sqrt{1 + \left(\frac{y}{2}\right)^2}\, dy = \int_2^4 \sqrt{1 + \frac{y^2}{4}}\, dy$$

(b) The surface area generated by revolving the section of the curve $y^2 - 4x = 0$ from $(1, 2)$ to $(4, 4)$ around the line $y = -2$.

Using the work from above, we know that $x' = \frac{y}{2}$ and $y' = \frac{1}{\sqrt{x}}$. The distance to the axis of rotation is $y + 2$ and $2\sqrt{x} + 2$, respectively. Therefore, the surface area is

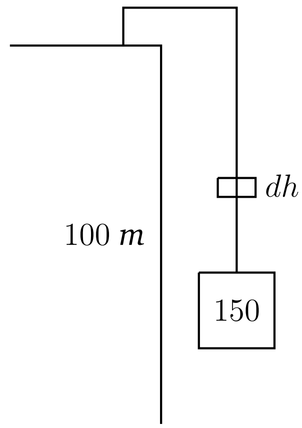
$$2\pi \int_1^4 (2\sqrt{x} + 2)\sqrt{1 + \frac{1}{x}}\, dx$$

OR

$$2\pi \int_2^4 (y + 2)\sqrt{1 + \frac{y^2}{4}}\, dy$$

23. (5 points) Set up the integrals to calculate—but do not evaluate—the following:

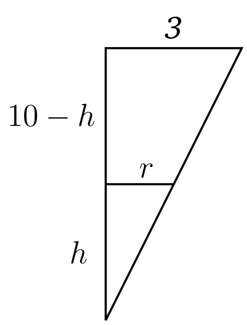
(a) The work to lift a 150 kg load 100 m using a steel chain that weighs 5 kg/m.



We have $\text{Mass}_{\text{load}} = 150$ kg and $\text{Weight}_{\text{load}} = 150$ g. Furthermore, We have $\text{Mass}_{\text{chain}} = 5\ dh$ and $\text{Weight}_{\text{chain}} = 5g\ dh$. Therefore, the work is

$$\int_0^{100} (150g + 5gh)\ dh$$

(b) The work to pump out a liquid with density 900kg/m^3 out of a conical tank, set up so that it is resting on a point, of height 10m with radius 3m that is filled to a depth of 8m.



Let $\rho = 900$ kg/m^3. We have $A = \pi r^2 = \pi \left(\frac{3}{10}h\right)^2 = \frac{9\pi}{100}h^2$. Now $V = A\ dh = \frac{9\pi}{100}h^2\ dh$. Furthermore, Mass $= \rho V = \frac{9\pi\rho}{100}h^2\ dh$ so that Weight $= Mg = \frac{9\pi\rho g}{100}h^2\ dh$. Finally, Work $= F \cdot d = \frac{9\pi\rho g}{100}h^2\ dh \cdot (10 - h)$ so that the total work is

$$\int_0^8 \frac{9\pi\rho g}{100}h^2(10 - h)\ dh = \frac{9\pi\rho g}{100}\int_0^8 h^2(10 - h)\ dh$$

24. (5 points) Solve the following differential equations:

$x^2y' + 3xy = 1$

Note that $\int \frac{3}{x}\,dx = 3\ln x = \ln x^3$ *and* $e^{\ln x^3} = x^3$. *This is the integrating factor.*

$$\begin{aligned}
x^2y' + 3xy &= 1 \\
x^3\left(x^2y' + 3xy\right) &= x^3 \\
x^3y' + 3x^2y &= x \\
\frac{d}{dx}(yx^3) &= x \\
\int \frac{d}{dx}(yx^3)\,dx &= \int x\,dx \\
yx^3 &= \frac{x^2}{2} + C \\
y &= \frac{1}{2x} + \frac{C}{x^3}
\end{aligned}$$

$y' - xe^y = 0$

$$\begin{aligned}
y' - xe^y &= 0 \\
y' &= xe^y \\
\frac{dy}{dx} &= xe^y \\
\int e^{-y}\,dy &= \int x\,dx \\
-e^{-y} &= \frac{x^2}{2} + C \\
-y &= \ln\left(C - \frac{x^2}{2}\right) \\
y &= -\ln\left(C - \frac{x^2}{2}\right) \\
y &= \ln\left(\frac{1}{C - \frac{x^2}{2}}\right) \\
y &= \ln\left(\frac{2}{C - x^2}\right)
\end{aligned}$$