Problem 1: Sequences

Determine if the following sequences converge; if so, find their limit:

(a) $a_n = \dfrac{5n^2 + n - 1}{2n^3 + 2n^2 - 5n - 2}$

(b) $b_n = \dfrac{3n^4 + 1}{10n^3 + 16n^2 + 35n + 29}$

(c) $c_n = \dfrac{2n^2 - n + 5}{3n^2 + n - 6}$

(d) $d_n = \dfrac{3^{n+4}}{5^{n-1}}$

(e) $e_n = \cos\left(\dfrac{n\pi}{2}\right)$

(f) $f_n = \ln(3n^5 - 2) - \ln(2n^5 + n - 2)$

(g) $g_n = \left(\dfrac{\pi}{4}\right)^n$

Problem 2: More Sequences

Find the limits of the following sequence, if they converge. You may want to make reference to the previous problem.

(a) $p_n = \left(\dfrac{\pi}{4}\right)^n + \dfrac{2n^2 - n + 5}{3n^2 + n - 6}$

(b) $q_n = \dfrac{2n^2 - n + 5}{3n^2 + n - 6} \cos\left(\dfrac{n\pi}{2}\right)$

(c) $r_n = \dfrac{3^{n+4}}{5^{n-1}} \cdot \dfrac{2n^2 - n + 5}{3n^2 + n - 6}$

(d) $s_n = \dfrac{3^{n+4}}{5^{n-1}} \cdot \cos\left(\dfrac{n\pi}{2}\right)$

(e) $t_n = \ln(3n^5 - 2) + \dfrac{5n^2 + n - 1}{2n^3 + 2n^2 - 5n - 2} + \dfrac{3^{n+4}}{5^{n-1}} \cdot \cos\left(\dfrac{n\pi}{2}\right) - \ln(2n^5 + n - 2)$

Problem 3: Divergence Test

For each of the sequences from Problem 1, form an infinite series using each of them. For example, the series for (a) would be $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2 + n - 1}{2n^3 + 2n^2 - 5n - 2}$. Of these newly formed series, to which do the Divergence Test apply? Explain.

Problem 4: The Theory of Infinite Series

(a) Does the following series converge: $\sum_{n=0}^{\infty}(-1)^n$

(b) Does the following series converge: $\sum_{n=0}^{\infty}(-1)^{n+1}$

(c) Does the following series converge: $\sum_{n=0}^{\infty}(-1)^n + (-1)^{n+1}$

(d) Is the following equality true?

$$\sum_{n=0}^{\infty}(-1)^n + (-1)^{n+1} \stackrel{?}{=} \sum_{n=0}^{\infty}(-1)^n + \sum_{n=0}^{\infty}(-1)^{n+1}$$

Explain why or why not.

(e) Using the above parts, explain when the following operations are valid:

$$\sum_{n=0}^{\infty} a_n + b_n = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$$
$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n + b_n$$

Problem 5: Geometric Series

Write the following series as a geometric series. Then determine if the series converges or diverges. If the series converges, find the sum. Be sure to justify why the series converges or diverges.

(a) $\frac{1}{2} + \frac{3}{10} + \frac{9}{50} + \frac{27}{250} + \frac{81}{1250} + \cdots$

(b) $-\frac{1}{2} + \frac{3}{4} - \frac{9}{8} + \frac{27}{16} - \frac{81}{32} + \cdots$

(c) $\sin(12) + \sin^3(12) + \sin^5(12) + \sin^7(12) + \sin^9(12) + \cdots$

(d) $1 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \sqrt[5]{2} + \cdots$

Find the sum of the following geometric series:

(e) $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$

(f) $\sum_{n=0}^{\infty} 3^{2-n}2^{n+1}$

(g) $\sum_{n=1}^{\infty} 3^{2-n}2^{n+1}$

Problem 6: Almost Geometric Series

Is the following sequence a geometric sequence? Explain.

$$\{s_n\}_{n=1}^{\infty} = \left\{ \frac{5}{6}, \frac{13}{36}, \frac{35}{216}, \frac{97}{1296}, \frac{275}{7776}, \cdots \right\}$$

Note that the sequence above was formed by the sequence

$$\left\{ \frac{1}{2} + \frac{1}{3}, \frac{1}{4} + \frac{1}{9}, \frac{1}{8} + \frac{1}{27}, \frac{1}{16} + \frac{1}{27}, \cdots \right\}$$

Writing the sequence like this, is the sequence geometric? By using two known geometric series, determine whether the series above converges or diverges. If it converges, find the sum.

Problem 7: Geometric Sequences with a variable

Write the following series in the form $\sum_{n=0}^{\infty} ar^n$.

$$2 - 4y^3 + 8y^6 - 16y^9 + 32y^{12} - 64y^{15} + \cdots$$

Determine which values of y make the series converge. Do the same for the following series:

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^{2n+1}}$$

Problem 8: Even More Geometric Series

Find the value of x in the following:

$$\sum_{n=1}^{\infty} \frac{(2x+1)^{(2n+1)/2}}{3^{n+2}} = \frac{1}{18}$$

Problem 9: Telescoping Series

By writing out a few partial sums, find the sun of the following series:

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$$

Problem 10: Combining Series

Find the sum of the following series:

$$\sum_{n=1}^{\infty} \left(3^{2-n} 2^{n+1} + \frac{1}{n^2+4n+3} \right)$$

Problem 11: Zeno's Paradox: Achilles and the Tortoise

Zeno of Elea (ca. 490–430 BC) gave a series of statements to support Parmenides' doctrine, which approximately stated that existence and change may be an illusion. Of his proposals, nine survived till the modern era. Though they are different, they can all be restated in similar ways so that they nearly all exemplify the same paradox. The most famous of Zeno's Paradox is that of the Achilles and the Tortoise, which we give as follows:

Achilles and a tortoise run a 100 meter race. Because Achilles is faster and a fair minded individual, he gives the tortoise a 50 meter head start. The race begins and once the tortoise reaches the 50 meter mark, Achilles begins to run. Because he is quick, he reaches the 50 meter mark in short time. But the tortoise has since moved and is still ahead. Achilles then runs to where the tortoise is. But because this took some time, the tortoise has since moved on. This process continues ad infinitum and hence Achilles never catches the tortoise – ever.

Aristotle in his *Physics* VI:9 phrased it this way, "In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead." The Greeks knew this conflicted with reality and appropriately labeled it a paradox. There have been many refutations, in many forms, of Zeno's paradox since its original statement. Thought it took over a millennium before it could be given serious refutation with – as it turns out – the theory of infinite series.

Let us put some numbers to this problem. Suppose Achilles runs 10 meters a second while the tortoise runs 1 meter per second. Using this set up Achilles clearly wins the race. When does he catch the tortoise? Where was the (at least mathematical) flaw in Zeno's logic?

Problem 12: A "Proof" of God's Existence

Dom Guido Grandi was an Italian monk from Cremona, Italy. When not performing his priestly duties, he worked on mathematics and engineering in is spare time. Among other things, he studied various curves, including the conical loxodrome and the versoria curve (which Fermat also studied). This curve was later studied by one of the great female mathematicians: Maria Agnesi. Through a mistranslation, the curve is now known as the "Witch of Agnesi." Grandi also corresponded with Leibniz and may have helped contribute to his later discovery of Calculus. By 1707, he was named as a court mathematician for the Grand Duke of Tuscany Cosimo III de Medici. During his life, Grandi gave the following "proof" of God's existence via the creation of something from nothing.

$$\begin{aligned}0 &= 0\\ &= (1-1)+(1-1)+(1-1)+\cdots\\ &= 1-1+1-1+1-1+\cdots\\ &= 1+(-1+1)+(-1+1)+\cdots\\ &= 1\end{aligned}$$

Of course given any number x, we have $x = 1 \cdot x$. But then the above implies $x = 1 \cdot x = 0 \cdot x = 0$. So every number is zero? Explain the error in this "proof."

We should put Grandi's error into historical context—less we judge him too harshly. It should be noted that during this time, many mathematicians were not concerned with the concept of convergence and Grandi's mathematics was not terribly uncommon in his time. Though many mathematicians avoided working with such objects because infinite objects or infinite operations were not yet well understood or even defined. It was during this time, the time of Newton and Leibniz, that infinitesimal quantities were more thoroughly defined.

It would not be until the 1600s and late 1700s straight through to the mid 1800s during the time of Euler, Cauchy, Weierstrass, and others that these concepts would be well understood. Though they still often did not work with them in rigorous ways. Eventually through work of, among others Cantor, it was shown that mathematicians needed a new level of rigor in proof writing and a standardizing of definitions. In the early 1900s, a group of mathematicians came together to gather, formalize, standardize, and unify mathematics, especially set theory and group theory. They published under the pseudonym Nicolas Bourbaki. Though this is all a very terse and rather exaggerative account of the dense history of mathematics 1600s to 1900s but it gives the gist of Grandi's mistake in its historical context. For a better and more modern example of "creation from nothing", one might read about the Banach-Tarski Paradox. This along with Grandi's "proof", show that extreme care need be taken when trying to translate the world of mathematics to "real world" problems and vice versa.