Radius/Interval of Convergence

Determine the interval and radius of convergence of each of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+2}$

(b) $\displaystyle\sum_{n=0}^{\infty} x^n \sqrt{n}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n 2^n}{n!\sqrt{n+1}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n x^n}{2^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n\,2^n}$

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{3^n \ln n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n x^n}{(2n)!}$

(h) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Deriving Taylor Series

Derive the Maclaurin series for each of the following functions. Determine the radius and interval of convergence of each.

(a) $\dfrac{1}{1-x}$

(b) $\sin x$

(c) $\arctan x$

(d) e^x

(e) $\ln(1+x)$

Quickly Deriving Taylor Series

Use a known Taylor series to find the Maclaurin series of the following functions. You may need to differentiate or integrate known series to find them!

(a) $\dfrac{1}{1+2x}$

(b) $\dfrac{x^3}{1-3x^2}$

(c) e^{2+x}

(d) $\sqrt{e^x}$

(e) x^2e^{-x}

(f) $\dfrac{1+x}{1-x}$

(g) $\ln\left(\dfrac{1-x}{1+x}\right)$

(h) $\dfrac{1}{(1-x)^2}$

(i) $e^x \sin x$

(j) $2\sin x \cos x$

Taylor Series and Limits

Use a Taylor series to evaluate the following limits:

(a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

(b) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x}$

(c) $\displaystyle\lim_{x\to 0} \frac{\arctan x - x}{x^3}$

Harder Taylor Series Limits (Big O)

Use a finite Taylor series approximation and its error term (the big O notation) to evaluate the following limits:

(a) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + 2x^3}{x^2}$

(b) $\displaystyle\lim_{x\to 0} \frac{\cos x \sin(x\ln(1+x))}{x^2}$

Taylor Series and Integration

Use a Taylor series to integrate the following:

(a) $\int x \cos x^2 \, dx$

(b) $\int \arctan x^2 \, dx$

(c) $\int \frac{e^x - 1}{x} \, dx$

Taylor Series and Integration Approximation

Use a Taylor series to approximate each of the following integrals. Determine the maximum error in your approximation.

(a) $\int_0^1 e^{-x^2} \, dx$

(b) $\int_0^1 \sin x^2 \, dx$

(c) $\int_0^1 \frac{\sin x}{x} \, dx$

(d) $\int_0^1 \frac{3}{x^3 + 27} \, dx$

Taylor Series and Approximation

Approximate each of the following using a Taylor series. Give the maximum error in your approximation.

(a) $\ln(1.1)$

(b) $\cos(1/3)$

(c) $\arctan(3/2)$

(d) $\sqrt{101}$

(e) $\sqrt[4]{e}$

Taylor Series and Infinite Series

Use a Taylor series to evaluate each of the following sums:

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(b) $\sum_{n=1}^{\infty} \frac{3^n}{5^n n!}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

(d) $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{36^n (2n)!}$

(e) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n3^n}$

(f) $\sum_{n=0}^{\infty} \frac{n}{2^n}$

Miscellaneous Problems

(a) Show that e is an irrational number using the following argument: Assume that e is rational. Let $e = p/q$. Approximate e using its Taylor series to the qth term with its error term. Show that this approximation can be written $s_q + \frac{e^w}{(q+1)!}$, where s_q is a sum of q numbers and $0 < w < 1$. Explain why $q!(e - s_q)$ is an integer. Explain why $q!(e - s_q) < 1$. Explain why this shows that e cannot be irrational.

(b) If $f(x)$ has a power series about $x = 0$ with radius of convergence R, show that $g(x) = f\left(\frac{x-1}{2}\right)$ has a power series with radius of convergence of $2R$ about $x = 1$.

(c) Use the Taylor series for $\cos x, \sin x$ to show that $e^{i\theta} = \cos\theta + i\sin\theta$, where $i^2 = -1$. Then use this to show that $e^{i\pi} + 1 = 0$.