Problem 1: Sequences

Determine the limit of the following sequences.

(a) $a_n = \dfrac{2n^2 - 3n + 1}{5n^6 + 2n + 1}$

(b) $b_n = \dfrac{2n - 3n^5 + 1}{n^3 + 5n + 7}$

(c) $c_n = \dfrac{5n^3 + 2n + 1}{7n^3 - n + 4}$

(d) $d_n = \sqrt[n]{5n}$

(e) $e_n = \left(1 + \dfrac{3}{7n}\right)^{4n/11}$ [Pun filled, no?]

(f) $f_n = n \sin\left(\dfrac{1}{n}\right)$

Problem 2: End Behavior

Determine if the following series converge or diverge. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

(b) $\displaystyle\sum_{n=1}^{\infty} \arctan n$

(c) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{2}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 + n + 2}{n^2 + 3n + 1}$

Problem 3: Telescoping Series

Determine if the following series converge or diverge. If the series converges, find the sum. If the series diverges, prove it.

(a) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

(d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right)$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+2} - \frac{1}{n}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+3}$

Problem 4: Geometric Series

Determine if the following series converges or diverges. If the series converges, find the sum. If the series diverges, prove it.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{e^{n+1}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{5^{n-1}}{\pi^{2n-1}}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{n+1} - 5^n}{\pi^n}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n-1} + 4^{n+2}}{5^{n+1}}$

(e) $\displaystyle\sum_{n=0}^{\infty} (\sin 3.4)^n$

Problem 5: Alternating Series

Determine if the following series conditionally converges, absolutely converges, or diverges. If the series converges, determine (at most) how many terms are needed to add to approximate the sum to three decimal digits of accuracy. If the series diverges, prove it.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{n}}$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

(e) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n^2}{10^n}$

(f) $\sum_{n=0}^{\infty} (-1)^n \frac{n^2+3}{n^3+4n+1}$

Problem 6: Comparison Tests

Determine if the following series converges or diverges. Be sure to justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{2}{n^3+2}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

(c) $\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$

(d) $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+5}$

(e) $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^3-1}$

(f) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$

(g) $\sum_{n=1}^{\infty} \frac{3n^2-2n-1}{n^3+n+1}$

(h) $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^4}$

(i) $\sum_{n=1}^{\infty} \frac{n \ln n}{n^4+5n+3}$

Problem 7: Root & Ratio Test

Determine if the following series converge or diverge. Be sure to be as specific as possible and justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{7^n}{n!\sqrt{n}}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^n}{(2n)!}$

(d) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3n^3+5n^2+n-7}{5n^3+6n+4}\right)^n$

(e) $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{(n+1)\sqrt{n-1}2^n}{5^n}$

Problem 8: Power Series

Find the center, interval of convergence, and radius of convergence of the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^n}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n2^n}{(2x-3)^n}$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(x+2)^n}{n2^n}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{\sqrt{n}}(x+2)^n$

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{(x-3)^n}$

(f) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^3 3^n}$

(g) $\displaystyle\sum_{n=1}^{\infty}\sqrt{n}x^n$

(h) $\displaystyle\sum_{n=1}^{\infty}\frac{2^{2n+1}x^n}{n!}$

Problem 9: Deriving Taylor Series

Derive the Maclaurin series for each of the following functions. Determine the radius and interval of convergence of each.

(a) $\dfrac{1}{1-x}$

(b) $\sin x$

(c) $\arctan x$

(d) e^x

(e) $\ln(1+x)$

Problem 10: Quickly Deriving Taylor Series

Use a known Taylor series to find the Maclaurin series of the following functions. You may need to differentiate or integrate known series to find them!

(a) $\dfrac{1}{1+2x}$

(b) $\dfrac{x^3}{1-3x^2}$

(c) e^{2+x}

(d) $\sqrt{e^x}$

(e) x^2e^{-x}

(f) $\dfrac{1+x}{1-x}$

(g) $\ln\left(\dfrac{1-x}{1+x}\right)$

(h) $\dfrac{1}{(1-x)^2}$

(i) $e^x \sin x$

(j) $2\sin x\cos x$

Problem 11: Taylor Series and Limits

Use a Taylor series to evaluate the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

(b) $\lim_{x \to 0} \frac{\cos x - 1}{x}$

(c) $\lim_{x \to 0} \frac{\arctan x - x}{x^3}$

Problem 12: Integration with Taylor Series

Use a Taylor series to integrate the following:

(a) $\int x \cos x^2 \, dx$

(b) $\int \arctan x^2 \, dx$

(c) $\int \frac{e^x - 1}{x} \, dx$

Problem 13: Taylor Series and Infinite Series

Use a Taylor series to evaluate each of the following sums:

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(b) $\sum_{n=1}^{\infty} \frac{3^n}{5^n n!}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

(d) $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{36^n (2n)!}$

(e) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n 3^n}$

(f) $\sum_{n=0}^{\infty} \frac{n}{2^n}$