Math 296: Exam 1 **Name:** *Caleb M^c Whorter — Solutions*

Fall – 2017

09/20/2017

80 Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains 12 pages (including this cover page) and 8 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	18	
2	10	
3	10	
4	10	
5	10	
6	10	
7	17	
8	15	
Total:	100	

1. (18 points) Integrate the following:

$$\int x^2 \left(4x + 5 - \frac{1}{\sqrt{x}}\right)\, dx$$

$$\begin{aligned}\int x^2 \left(4x + 5 - \frac{1}{\sqrt{x}}\right)\, dx &= \int \left(4x^3 + 5x^2 - \frac{x^2}{\sqrt{x}}\right)\, dx \\ &= \int \left(4x^3 + 5x^2 - x^{3/2}\right)\, dx \\ &= \boxed{x^4 + \frac{5}{3}x^3 - \frac{2}{5}x^{5/2} + C}\end{aligned}$$

$$\int_3^5 \sqrt{(2x-6)^3}\, dx$$

$$\begin{aligned}u &= 2x - 6 \\ du &= 2\ dx \Longleftrightarrow dx = \frac{du}{2}\end{aligned}$$

If $x = 3$, then $u = 2(3) - 6 = 0$. If $x = 5$, then $u = 2(5) - 6 = 4$. Therefore,

$$\int_3^5 \sqrt{(2x-6)^3}\, dx = \int_0^4 \sqrt{u^3}\, \frac{du}{2} = \frac{1}{2}\int u^{3/2}\, du = \frac{1}{2} \cdot \frac{2}{5}u^{5/2}\bigg|_0^4 = \frac{1}{5} \cdot \left(4^{5/2} - 0^{5/2}\right) = \boxed{\frac{32}{5}}$$

$$\int \frac{x+3}{x-1}\,dx$$

$$\begin{aligned} u = x-1 &\iff x = u+1 \\ du &= dx \end{aligned}$$

$$\int \frac{x+3}{x-1}\,dx = \int \frac{u+1+3}{u}\,du = \int \left(1+\frac{4}{u}\right)\,du = \; u \,+\, 4\ln|u| \,+\, C = \boxed{x-1+4\ln|x-1|+C}$$

$$= \boxed{x+4\ln|x-1|+C}$$

$$\int \frac{2}{4x^2+3}\,dx$$

$$\int \frac{2}{4x^2+3}\,dx = 2\int \frac{dx}{4x^2+3} = 2\int \frac{dx}{4x^2+3}\cdot\frac{1/3}{1/3} = 2\int \frac{1/3}{\frac{4}{3}x^2+1}\,dx = \frac{2}{3}\int \frac{dx}{(2/\sqrt{3}x)^2+1}$$

$$\begin{aligned} u &= \frac{2}{\sqrt{3}}x \\ du &= \frac{2}{\sqrt{3}}\,dx \iff dx = \frac{\sqrt{3}}{2}\,du \end{aligned}$$

$$\frac{2}{3}\int \frac{dx}{(2/\sqrt{3}x)^2+1} = \frac{2}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{du}{u^2+1} = \frac{\sqrt{3}}{3}\arctan u + C = \boxed{\frac{\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}x\right)+C}$$

$$= \boxed{\frac{1}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x\right)+C}$$

2. (10 points) Integrate the following:

$$\int x^4 \ln x \, dx$$

$\ln x$	$\dfrac{x^5}{5}$
$\dfrac{1}{x}$	x^4

$$\int x^4 \ln x \, dx = \frac{1}{5}x^5 \ln x - \int \frac{x^5}{5x} \, dx = \frac{1}{5}x^5 \ln x - \frac{1}{5}\int x^4 \, dx = \boxed{\frac{1}{5}x^5 \ln x - \frac{x^5}{25} + C}$$

$$= \frac{5x^5 \ln x - x^5}{25} + C$$

$$= \boxed{\frac{x^5(5\ln x - 1)}{25} + C}$$

$$\int \arctan x \, dx$$

$\arctan x$	x
$\dfrac{1}{1+x^2}$	1

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} \, dx = \boxed{x \arctan x - \frac{1}{2}\ln|x^2+1| + C}$$

3. (10 points) Integrate the following:

$$\int x^3 e^{2x}\, dx$$

u		dv
x^3	$+$	e^{2x}
$3x^2$	$-$	$\frac{e^{2x}}{2}$
$6x$	$+$	$\frac{e^{2x}}{4}$
6	$-$	$\frac{e^{2x}}{8}$
0		$\frac{e^{2x}}{16}$

$$\int x^3 e^{2x}\, dx = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{6}{8}xe^{2x} - \frac{6}{16}e^{2x} + C$$

$$= \boxed{\frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C}$$

$$= \frac{4}{8}x^3e^{2x} - \frac{6}{8}x^2e^{2x} + \frac{6}{8}xe^{2x} - \frac{3}{8}e^{2x} + C$$

$$= \boxed{\frac{e^{2x}}{8}\left(4x^3 - 6x^2 + 6x - 3\right) + C}$$

4. (10 points) Integrate the following:

$$\int e^{x/2} \sin x \, dx$$

u	sign	dv
$\sin x$	$+$	$e^{x/2}$
$\cos x$	$-$	$2e^{x/2}$
$-\sin x$	$+$	$4e^{x/2}$

$$\int e^{x/2} \sin x \, dx = 2e^{x/2} \sin x - 4e^{x/2} \cos x - \int 4e^{x/2} \sin x \, dx$$

$$\int e^{x/2} \sin x \, dx = 2e^{x/2} \sin x - 4e^{x/2} \cos x - 4 \int e^{x/2} \sin x \, dx$$

$$5 \int e^{x/2} \sin x \, dx = 2e^{x/2} \sin x - 4e^{x/2} \cos x$$

$$\int e^{x/2} \sin x \, dx = \boxed{\frac{2e^{x/2} \sin x - 4e^{x/2} \cos x}{5} + C}$$

$$\int e^{x/2} \sin x \, dx = \boxed{\frac{2e^{x/2}}{5} (\sin x - 2 \cos x) + C}$$

5. (10 points) Integrate the following:

$$\int \tan^2 x \ \sec^4 x \, dx$$

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\int \tan^2 x \ \sec^4 x \, dx = \int \tan^2 x \sec^2 x \cdot \sec^2 x \, dx$$

$$= \int \tan^2 x(\tan^2 x + 1) \cdot \sec^2 x \, dx$$

$$u = \tan x$$

$$du = \sec^2 x \, dx$$

$$\int \tan^2 x \ \sec^4 x \, dx = \int \tan^2 x(\tan^2 x + 1) \cdot \sec^2 x \, dx$$

$$= \int u^2(u^2 + 1) \, du$$

$$= \int (u^4 + u^2) \, du$$

$$= \frac{u^5}{5} + \frac{u^3}{3} + C$$

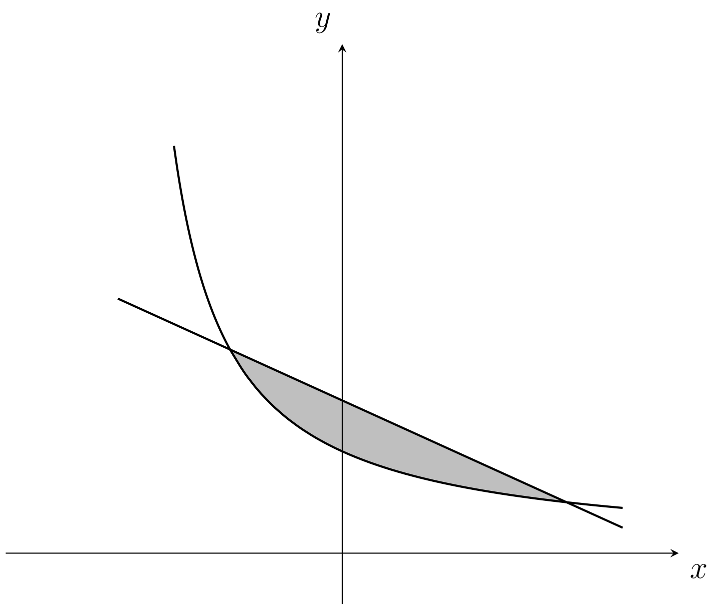
$$= \boxed{\frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C}$$

6. (10 points) Set up, as completely as possible, ***but do not evaluate*** an integral that can be used to calculate the arc length of the curve $f(x) = x^2 \sin x$ for $0 \le x \le \pi$.

$$f(x) = x^2 \sin x$$
$$f'(x) = x^2 \cos x + 2x \sin x$$

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx = \boxed{\int_0^\pi \sqrt{1 + (x^2 \cos x + 2x \sin x)^2}\, dx}$$

7. (17 points) Throughout this problem, let $f(x) := 3 - x$ and $g(x) := \dfrac{4}{x+2}$. Both $f(x)$ and $g(x)$ are plotted below.



(a) Set up ***but do not evaluate*** an integral to compute the area bound by $f(x)$ and $g(x)$.

$$\begin{aligned} f(x) &= g(x) \\ 3 - x &= \frac{4}{x+2} \\ (3-x)(x+2) &= 4 \\ -x^2 + x + 6 &= 4 \\ 0 &= x^2 - x - 2 \\ 0 &= (x-2)(x+1) \end{aligned}$$

Therefore, $x = -1, 2$.

$$\boxed{A = \int_{-1}^{2} \left((3-x) - \frac{4}{x+2} \right) \, dx}$$

Or of course, one could do

$$\boxed{A = \int_{1}^{4} \left((3-y) - \frac{4-2y}{y} \right) \, dy}$$

(b) Set up as completely as possible *both* the integral for the Dish/Washer Method and the integral for the Shell method used to calculate the volume of solid formed by rotating the region bound by $f(x)$ and $g(x)$ about the line $x = -4$. Be sure to label each integral. ***Do not evaluate these integrals.***

Note that $y = 3 - x \Longleftrightarrow x = 3 - y$ *and* $y = \frac{4}{x+2} \Longleftrightarrow x = \frac{4}{y} - 2 = \frac{4-2y}{y}$.

Disks: $$\pi \int_1^4 \left((3-y) - (-4)\right)^2 - \left(\frac{4-2y}{y} - (-4)\right)^2 \, dy$$

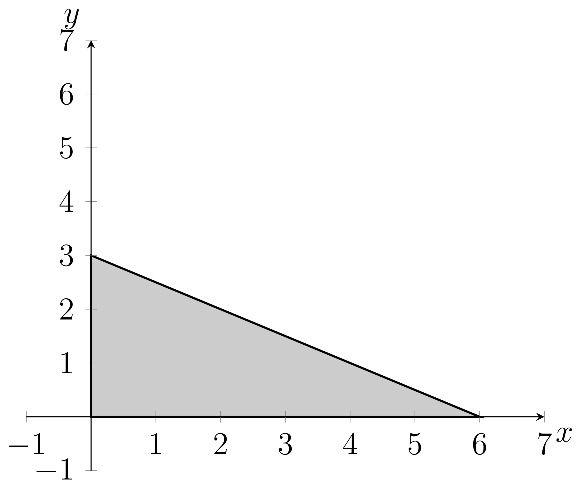
Shells: $$2\pi \int_{-1}^2 (x - (-4)) \left((3-x) - \left(\frac{4}{x+2}\right)\right) \, dx$$

(c) Set up as completely as possible *both* the integral for the Dish/Washer Method and the integral for the Shell method used to calculate the volume of solid formed by rotating the region bound by $f(x)$ and $g(x)$ about the line $y = -2$. Be sure to label each integral. ***Do not evaluate these integrals.***

Disks: $$\pi \int_{-1}^2 \left(2 + (3-x)\right)^2 - \left(2 + \frac{4}{x+2}\right)^2 \, dx$$

Shells: $$2\pi \int_1^4 (2+y) \left((3-y) - \frac{4-2y}{y}\right) \, dy$$

8. (15 points) Let R be the region bound by the line $y = \dfrac{6-x}{2}$, the y–axis, and the x–axis.



If S is a solid whose base is the region R, set up ***but do not evaluate*** integrals to calculate the volume of S if...

(a) slices perpendicular to the x–axis are squares.

$$V = \int A(x)\,dx = \int s^2\,dx = \boxed{\int_0^6 \left(\frac{6-x}{2}\right)^2 dx}$$

(b) slices perpendicular to the y–axis are semicircles.

$$y = \frac{6-x}{2} \Longleftrightarrow x = 6 - 2y$$

$$V = \int A(y)\,dy = \int \frac{1}{2}\pi r^2\,dy = \frac{\pi}{2}\int \left(\frac{d}{2}\right)^2\,dy = \frac{\pi}{8}\int d^2\,dy = \boxed{\frac{\pi}{8}\int_0^3 (6-2y)^2\,dy}$$

(c) slices perpendicular to the x–axis are 30° – 60° – 90° triangles with the shortest leg lying in the region R. [Recall in a 30° – 60° – 90° triangle, the sides are in ratio $1 : \sqrt{3} : 2$.]

The base b *is the shortest side and is lying in the plane. Given the ratios in a* 30° – 60° – 90° *triangle,* $\sqrt{3}b = h$.

$$V = \int A(x)\,dx = \int \frac{1}{2}bh\,dx = \frac{1}{2}\int b(\sqrt{3}b)\,dx = \frac{\sqrt{3}}{2}\int b^2\,dx = \boxed{\frac{\sqrt{3}}{2}\int_0^6 \left(\frac{6-x}{2}\right)^2\,dx}$$