Math 296: Exam 2
Fall – 2017
10/18/2017
80 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 10 pages (including this cover page) and 9 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	15	
9	15	
Total:	100	

1. (10 points) Set up completely as possible *but do not evaluate* an integral to calculate the surface area of the surface generated by revolving the curve $f(x) = x^3$, $-1 \le x \le 1$, about the line...

   (a) $x = -3$.

$$f(x) = x^3 \implies f'(x) = 3x^2 \ \wedge \ f'(x)^2 = 9x^4$$

$$\boxed{2\pi \int_{-1}^{1} (x+3)\sqrt{1+9x^4}\ dx}$$

   (b) $y = -3$.

$$\boxed{2\pi \int_{-1}^{1} (x^3+3)\sqrt{1+9x^4}\ dx}$$

2. (10 points) Integrate the following:

$$\int_0^\infty \frac{x}{(x^2+1)^2}\,dx$$

We first treat this as an indefinite integral. Consider the following substitution:

$$u = x^2 + 1$$
$$du = 2x\,dx \iff dx = \frac{du}{2x}$$

This substitution yields...

$$\int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int \frac{du}{u^2} = \frac{-1}{2u} + C = \frac{-1}{2(x^2+1)} + C$$

We must then have

$$\begin{aligned}\int_0^\infty \frac{x}{(x^2+1)^2}\,dx &:= \lim_{b\to\infty}\int_0^b \frac{x}{(x^2+1)^2}\,dx \\ &= \lim_{b\to\infty} \frac{-1}{2(x^2+1)}\Bigg|_0^b \\ &= \lim_{b\to\infty} \frac{-1}{2(b^2+1)} - \frac{-1}{2(0^2+1)} \\ &= 0 - \frac{-1}{2} \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

3. (10 points) Integrate the following:

$$\int \frac{4x^2+3x-10}{x^3+5x^2}\,dx$$

First, we calculate the partial fraction decomposition of the integrand:

$$\begin{aligned}\frac{4x^2+3x-10}{x^3+5x^2} = \frac{4x^2+3x-10}{x^2(x+5)} &= \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+5}\\ &= \frac{Ax(x+5)+B(x+5)+Cx^2}{x^2(x+5)}\\ &= \frac{Ax^2+5Ax+Bx+5B+Cx^2}{x^2(x+5)}\\ &= \frac{(A+C)x^2+(5A+B)x+5B}{x^2(x+5)}\end{aligned}$$

Relating the numerators, we obtain the system of equations:

$$\begin{aligned} x^2: &\quad A+C=4\\ x: &\quad 5A+B=3\\ 1: &\quad 5B=-10\end{aligned}$$

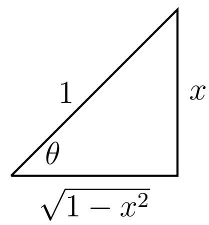
The last equation gives $B=-2$*. But then* $5A+B=5A-2$ *so that* $5A-2=3$ *and* $A=1$*. Finally, we must then have* $A+C=1+C$ *and* $1+C=4$ *so that* $C=3$*. Therefore,*

$$\begin{aligned}\int \frac{4x^2+3x-10}{x^3+5x^2}\,dx &= \int\left(\frac{1}{x}-\frac{2}{x^2}+\frac{3}{x+5}\right)\,dx\\ &= \boxed{\ln|x|+\frac{2}{x}+3\ln|x+5|+K}\\ &= \boxed{\frac{2}{x}+\ln|x(x+5)^3|+K}\end{aligned}$$

4. (10 points) Integrate the following:

$$\int \sqrt{1-x^2}\,dx$$

$$\begin{aligned} a^2+b^2 &= c^2 \\ b^2 &= \underbrace{c^2-a^2}_{1-x^2} \end{aligned}$$



$$\begin{aligned} \sin\theta &= \frac{x}{1} \\ x &= \sin\theta \\ dx &= \cos\theta\,d\theta \end{aligned}$$

$$\begin{aligned} \cos\theta &= \frac{\sqrt{1-x^2}}{1} \\ \sqrt{1-x^2} &= \cos\theta \end{aligned}$$

$$\begin{aligned} \int \sqrt{1-x^2}\,dx &= \int \cos\theta\cdot\cos\theta\,d\theta \\ &= \int \cos^2\theta\,d\theta \\ &= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta \\ &= \frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+C \\ &= \frac{\theta}{2}+\frac{\sin 2\theta}{4}+C \\ &= \frac{\theta}{2}+\frac{2\sin\theta\cos\theta}{4}+C \\ &= \frac{\theta}{2}+\frac{1}{2}\sin\theta\cos\theta+C \\ &= \frac{\theta+\sin\theta\cos\theta}{2}+C \end{aligned}$$

Therefore, we must have

$$\int \sqrt{1-x^2}\,dx = \boxed{\frac{\sin^{-1}x}{2}+\frac{x\sqrt{1-x^2}}{2}+C}$$

5. (10 points) Integrate the following:

$$\int_1^\infty \frac{dx}{2x^2+x}$$

We first treat this as an indefinite integral:

$$\int \frac{dx}{2x^2+x} = \int \frac{dx}{x(2x+1)}$$

First, we find the partial fraction decomposition of the integrand.

$$\frac{1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$$

Heaviside's method yields:

$$A = \frac{1}{2(0)+1} = 1; \qquad B = \frac{1}{-1/2} = -2$$

$$\int \frac{dx}{x(2x+1)} = \int \left(\frac{1}{x} - \frac{2}{2x+1}\right)\, dx = \ln|x| - \ln|2x+1| + C = \ln\left|\frac{x}{2x+1}\right| + C$$

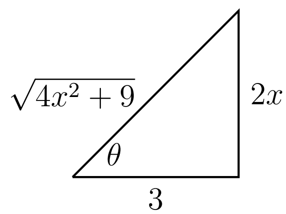
Therefore, we have

$$\begin{aligned}
\int_1^\infty \frac{dx}{2x^2+x} &:= \lim_{b\to\infty} \int_1^b \frac{dx}{2x^2+x} \\
&= \lim_{b\to\infty} \ln\left|\frac{x}{2x+1}\right|\Bigg|_1^b \\
&= \lim_{b\to\infty} \ln\left|\frac{b}{2b+1}\right| - \ln\left|\frac{1}{2+1}\right| \\
&= \ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{3}\right) \\
&= \ln\left(\frac{1}{2}\right) + \ln 3 \\
&= \boxed{\ln\left(\frac{3}{2}\right)}
\end{aligned}$$

6. (10 points) Integrate the following:

$$\int \frac{dx}{\sqrt{4x^2+9}}$$

$$\underbrace{a^2+b^2}_{4x^2+9} = c^2$$



$$\tan\theta = \frac{2x}{3}$$
$$x = \frac{3}{2}\tan\theta$$
$$dx = \frac{3}{2}\sec^2\theta\ d\theta$$

$$\sec\theta = \frac{\sqrt{4x^2+9}}{3}$$
$$\sqrt{4x^2+9} = 3\sec\theta$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{4x^2+9}} &= \int \frac{\frac{3}{2}\sec^2\theta}{3\sec\theta}\ d\theta \\
&= \frac{1}{2}\int \sec\theta\ d\theta \\
&= \frac{1}{2}\ln|\sec\theta+\tan\theta| + C \\
&= \boxed{\frac{1}{2}\ln\left|\frac{\sqrt{4x^2+9}}{3}+\frac{2x}{3}\right| + C} \\
&= \frac{1}{2}\ln\left|\frac{1}{3}\left(\sqrt{4x^2+9}+2x\right)\right| + C \\
&= \frac{1}{2}\ln\left|2x+\sqrt{4x^2+9}\right| + \frac{1}{2}\ln\left(\frac{1}{3}\right) + C \\
&= \boxed{\frac{1}{2}\ln\left|2x+\sqrt{4x^2+9}\right| + C}
\end{aligned}$$

7. (10 points) Integrate the following:

$$\int \frac{2x+4}{x^3+4x}\,dx$$

First, we find the partial fraction decomposition of the integrand:

$$\begin{aligned}
\frac{2x+4}{x^3+4x} = \frac{2x+4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx+C}{x^2+4} \\
&= \frac{A(x^2+4)+x(Bx+C)}{x(x^2+4)} \\
&= \frac{Ax^2+4A+Bx^2+Cx}{x(x^2+4)} \\
&= \frac{(A+B)x^2+Cx+4A}{x(x^2+4)}
\end{aligned}$$

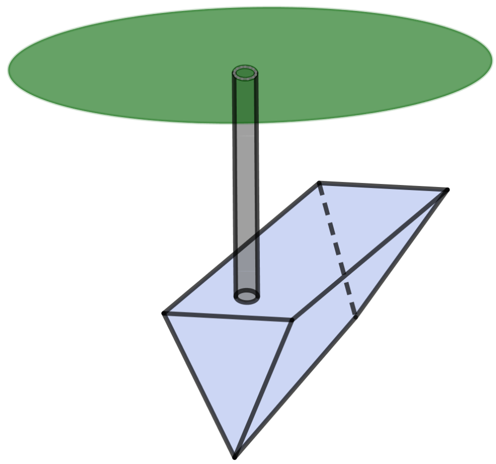
Relating the numerators, we obtain the system of equations:

$$\begin{aligned}
x^2: &\quad A+B=0 \\
x: &\quad C=2 \\
1: &\quad 4A=4
\end{aligned}$$

It is immediate that $C=2$ *and that* $A=1$. *As* $A+B=0$, $B=-A=-1$. *Therefore,*

$$\begin{aligned}
\int \frac{2x+4}{x^3+4x}\,dx &= \int \left(\frac{1}{x} + \frac{-x+2}{x^2+4}\right)\,dx \\
&= \int \left(\frac{1}{x} + \frac{-x}{x^2+4} + \frac{2}{x^2+4}\right)\,dx \\
&= \int \left(\frac{1}{x} + \frac{-x}{x^2+4} + \frac{2/4}{x^2/4+1}\right)\,dx \\
&= \int \left(\frac{1}{x} + \frac{-x}{x^2+4} + \frac{1}{2}\cdot\frac{1}{\left(\frac{x}{2}\right)^2+1}\right)\,dx \\
&= \boxed{\ln|x| - \frac{1}{2}\ln|x^2+4| + \arctan\left(\frac{x}{2}\right) + K} \\
&= \boxed{\ln\left|\frac{x}{\sqrt{x^2+4}}\right| + \arctan\left(\frac{x}{2}\right) + K}
\end{aligned}$$

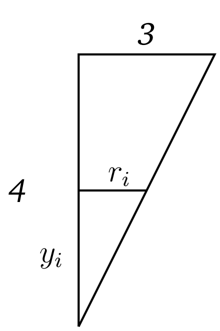
8. (15 points) Shown below is a storage tank with top width 6 m, maximum depth 4 m, and length 10 m. There is a 15 m tube connecting the tank to the surface. The tank is nearly filled with only a 1 m gap between the liquids surface and the top of the tank. Assuming the tank is filled with a liquid of density ρ, find an integral expression which would calculate the amount of work required to pump the contents of the tank to the surface. *Your integral expression should be as complete as possible but you do not need to evaluate your expression.*



Place the origin at the bottom and end of the tank (the bottom-most point in the figure) and orient the $+y$–axis upwards. Consider 'slicing' the tank into many, say n, thin slices with thickness Δy. If these slices are thin enough, we obtain approximately rectangular prisms whose volumes are $V = lwh$. Now consider the ith slab at height y_i above the bottom of the tank. How much work is done moving it to the top? The density of the liquid is ρ and we know $\rho = \frac{m}{V}$ so that $m = \rho V = \rho lwh$. This slab has to be moved $4 - y_i$ meters to reach the top of the tank and another 15 m to reach the surface—a total distance of $19 - y_i$ meters. [Note that $g \approx 9.8$ m/s^2.]

$$W_i = F_i \cdot d_i = m_i g = \rho l_i w_i h_i g \cdot d_i = \rho l_i w_i h_i g (19 - y_i)$$

We only need find l_i, w_i, and h_i for this slice. All slices have length 10 m. By similar triangles, we have...



$$\frac{r_i}{y_i} = \frac{3}{4}$$

$$r_i = \frac{3y_i}{4}$$

$$2r_i = \frac{3y_i}{2}$$

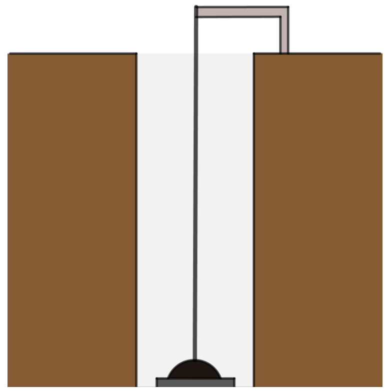
Then $w_i = 2r_i = \frac{3y_i}{2}$, $l_i = 10$, and $h_i = \Delta y$. Then we have

$$W \approx \sum_{i=1}^{n} W_i = \rho 10 \frac{3y_i}{2} \Delta y g (19 - y_i) = 15\rho g \sum_{i=1}^{n} y_i (19 - y_i) \Delta y$$

Therefore, we have

$$W = \lim_{\Delta \to 0} 15\rho g \sum_{i=1}^{n} y_i (19 - y_i) \Delta y = \boxed{15\rho g \int_0^3 y(19 - y)\, dy}$$

9. (15 points) A large drill ($18 \cdot 10^6$ kg) used in industrial mineral collection is sitting at the bottom of a mineshaft 45 m below the Earth's surface. The drill needs to be raised to the surface for repairs. If a thick steel cable weighing 50 kg/m is used to lift the drill, set up as completely as possible *but do not evaluate* an integral which would determine the work done lifting the drill to the surface.



Put the origin at the location of the drill with $+y$–axis oriented upwards. We consider the drill and chain separately. For the drill, this is a simple matter since the weight of the drill is constant and it is moved the 45 m to the surface:

$$W_{\textit{drill}} = F_{\textit{drill}} \cdot d_{\textit{drill}} = m_{\textit{drill}} g \cdot d_{\textit{drill}} = 45 g m_{\textit{drill}}$$

For the chain, consider breaking the chain into many small pieces, say n of them, with tiny length Δy. Clearly, we have

$$W_{\textit{chain}} \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} F_i \cdot d_i = \sum_{i=1}^{n} m_i g \cdot d_i$$

Consider the ith piece of chain at height y_i above the bottom of the mineshaft. This piece has linear density ρ and length Δy so that $m_i = \rho \Delta y$. This piece is at height y above the bottom so it must be moved $45 - y_i$ meters to reach to top of the mine shaft. But then we have $W_{\textit{chain}} \approx \sum_{i=1}^{n} \rho g \Delta y \cdot (45 - y_i)$. Therefore,

$$W_{\textit{chain}} = \lim_{\Delta \to 0} \sum_{i=1}^{n} \rho g \Delta y (45 - y_i) = \rho g \int_0^{45} 45 - y \, dy$$

The total work is then

$$W_{\textit{total}} = W_{\textit{drill}} + W_{\textit{chain}} = \boxed{45 g m_{\textit{drill}} + \rho g \int_0^{45} 45 - y \, dy}$$

Note that since $45 g m_{\textit{drill}} = \int_0^{45} g m_{\textit{drill}} \, dy$, this can also be expressed as a single integral by writing

$$\boxed{\int_0^{45} \left[g m_{\textit{drill}} + \rho g (45 - y) \right] dy}$$

where we have $m_{\textit{drill}} = 10 \cdot 10^6$ kg and $g \approx 9.8$ m/s^2.