Math 296: Exam 3 **Name:** *Caleb M^cWhorter — Solutions*

Fall – 2017

11/17/2017

80 Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains 9 pages (including this cover page) and 8 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. (10 points) Determine if the following series converge or diverge. If the series converges, find the sum. If the series diverges, prove it.

$$\sum_{n=0}^{\infty} 7\left(\frac{\pi}{2}\right)^n$$

This is a geometric series with $r = \frac{\pi}{2} > 1$. *Therefore, the series diverges. Alternatively,* $\lim_{n\to\infty} 7\left(\frac{\pi}{2}\right)^n = \infty$ *and therefore the series diverges by the Divergence Test. Both the Root and Ratio Test also apply and give divergence.*

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^{3n+1}}$$

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^{3n+1}} = \sum_{n=1}^{\infty} \frac{3^n \cdot 3^{-1}}{2^{3n} \cdot 2} = \sum_{n=1}^{\infty} \frac{1}{6} \cdot \frac{3^n}{8^n} = \sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{3}{8}\right)^n$$

This series is geometric with $r = \frac{3}{8} < 1$. *Therefore, the series converges. The sum of this series is*

$$\frac{\frac{1}{6} \cdot \frac{3}{8}}{1 - \frac{3}{8}} = \frac{\frac{1}{16}}{\frac{5}{8}} = \frac{1}{16} \cdot \frac{8}{5} = \boxed{\frac{1}{10}}$$

2. (10 points) Determine if the following series converges or diverges. Be sure to justify your answer completely.

$$\sum_{n=1}^{\infty} \left(\frac{6n-1}{5n+3} \right)^n$$

$$\lim_{n\to\infty} \left[\left(\frac{6n-1}{5n+3} \right)^n \right]^{1/n} = \lim_{n\to\infty} \frac{6n-1}{5n+3} = \frac{6}{5} > 1$$

Therefore, $\sum_{n=1}^{\infty} \left(\frac{6n-1}{5n+3} \right)^n$ *diverges by the Root Test.*

3. (10 points) Determine if the following series converge or diverge. Be sure to justify your answer completely.

$$\sum_{n=0}^{\infty} \sin\left(\frac{\pi n}{6n+1}\right)$$

$$\lim_{n\to\infty} \sin\left(\frac{\pi n}{6n+1}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \neq 0$$

Therefore, $\sum_{n=0}^{\infty} \sin\left(\frac{\pi n}{6n+1}\right)$ *diverges by the Divergence Test.*

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$$

$$\lim_{n\to\infty} \frac{\sin(1/\sqrt{n})}{1/\sqrt{n}} = 1$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *diverges by the* p*-test. Therefore,* $\sum_{n=0}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$ *diverges by the Limit Comparison Test.*

4. (10 points) Determine if the following series converges or diverges. If the series converges, find the sum. If the series diverges, prove it. [Hint: $\frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}$.]

$$\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)}$$

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} &= \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n+1}\right] \\
&= \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \cdots \\
&= 1 + \frac{1}{2} \\
&= \boxed{\frac{3}{2}}
\end{aligned}$$

OR

$$\begin{aligned}
S_N &= \sum_{n=2}^{N} \frac{2}{(n-1)(n+1)} = \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n+1}\right] \\
&= \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{N-3}} - \cancel{\frac{1}{N-1}}\right) + \left(\cancel{\frac{1}{N-2}} - \frac{1}{N}\right) + \left(\cancel{\frac{1}{N-1}} - \frac{1}{N+1}\right) \\
&= \frac{3}{2} - \frac{1}{N} - \frac{1}{N+1}
\end{aligned}$$

Therefore, we must have...

$$\sum_{n=2}^{\infty} \frac{2}{(n-1)(n+1)} := \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{3}{2} - \frac{1}{N} - \frac{1}{N+1}\right) = \frac{3}{2}$$

5. (10 points) Determine if the following series converge or diverge. Be sure to justify your answers completely.

$$\sum_{n=1}^{\infty} \frac{n^3+2n-1}{n^5+4}$$

$$\sum_{n=1}^{\infty} \frac{n^3+2n-1}{n^5+4} < \sum_{n=1}^{\infty} \frac{n^3+2n^3}{n^5} = 3\sum_{n=1}^{\infty} \frac{1}{n^2}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the* p*-test. Therefore,* $\sum_{n=1}^{\infty} \frac{n^3+2n-1}{n^5+4}$ *converges by the Comparison Test.*

OR

$$\lim_{n\to\infty} \frac{\frac{n^3+2n-1}{n^5+4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^5+2n^3-n^2}{n^5+4} = 1 < \infty$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the* p*-test. Therefore,* $\sum_{n=1}^{\infty} \frac{n^3+2n-1}{n^5+4}$ *converges by the Limit Comparison Test.*

$$\sum_{n=1}^{\infty} \frac{n^2+3}{\sqrt{2n^5-1}}$$

$$\sum_{n=1}^{\infty} \frac{n^2+3}{\sqrt{2n^5-1}} > \sum_{n=1}^{\infty} \frac{n^2}{\sqrt{2n^5}} = \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^5}} = \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *diverges by the* p*-test. Therefore,* $\sum_{n=1}^{\infty} \frac{n^2+3}{\sqrt{2n^5-1}}$ *diverges by the Comparison Test.*

OR

$$\lim_{n\to\infty} \frac{\frac{n^2+3}{\sqrt{2n^5-1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}\,(n^2+3)}{\sqrt{2n^5-1}} = \lim_{n\to\infty} \sqrt{\frac{n(n^2+3)^2}{2n^5-1}} = \lim_{n\to\infty} \sqrt{\frac{n^5+6n^3+9n}{2n^5-1}} = \sqrt{\frac{1}{2}} < \infty$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *diverges by the* p*-test. Therefore,* $\sum_{n=1}^{\infty} \frac{n^2+3}{\sqrt{2n^5-1}}$ *diverges by the Limit Comparison Test.*

6. (10 points) Determine if the following series is divergent, conditionally convergent, or absolutely convergent. Be sure to justify your answer completely.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}}$$

We have $\lim_{n\to\infty} \frac{1}{n\sqrt{\ln n}} = 0$ *and the sequence* $\frac{1}{n\sqrt{\ln n}}$ *is decreasing. Therefore, the series* $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}}$ *converges by the Alternating Series Test. Now consider the series*

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

Observe that $\frac{1}{x\sqrt{\ln x}}$ *is a positive, continuous, and decreasing function of* x. *Therefore, the Integral Test applies.*

$$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} := \lim_{b\to\infty} \int_2^{b} \frac{dx}{x\sqrt{\ln x}} \qquad u := \ln x, \quad du = \frac{dx}{x} \Rightarrow xdu = dx$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{du}{\sqrt{u}}$$

$$= \lim_{b\to\infty} 2\sqrt{u}\,\Big|_{\ln 2}^{b}$$

$$= \lim_{b\to\infty} 2\sqrt{b} - 2\sqrt{\ln 2}$$

$$= \infty$$

Therefore, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ *diverges by the Integral Test.*

Therefore, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}}$ *converges conditionally.*

7. (10 points) Determine if the following series converges or diverges. Be sure to justify your answer completely.

$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!}$$

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{((n+1)!)^3}{(3(n+1))!} \cdot \frac{(3n)!}{(n!)^3} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)!(n+1)!(n+1)!}{n!\,n!\,n!} \cdot \frac{(3n)!}{(3n+3)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)(n+1)(n+1)}{1} \cdot \frac{(3n)!}{(3n+3)(3n+2)(3n+1)(3n)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)(n+1)(n+1)}{(3n+3)(3n+2)(3n+1)} \right| \\
&= \frac{1}{27} < 1
\end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!}$ *converges absolutely by the Ratio Test.*

8. (10 points) Find the center, radius of convergence, and interval of convergence for the following power series:
$$\sum_{n=0}^{\infty}(-1)^n \frac{(x+3)^n}{n\,3^n}$$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n\,3^n}{(x+3)^n}\right| &= \lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{(x+3)^n}\cdot\frac{n}{n+1}\cdot\frac{3^n}{3^{n+1}}\right| \\
&= \lim_{n\to\infty}\left|(x+3)\cdot\frac{n}{n+1}\cdot\frac{1}{3}\right| \\
&= \left|\frac{x+3}{3}\right|
\end{aligned}$$

$$\begin{aligned}
\left|\frac{x+3}{3}\right| &< 1 \\
-1 < \frac{x+3}{3} &< 1 \\
-3 < x+3 &< 3 \\
-6 < x &< 0
\end{aligned}$$

$$R = \frac{0-(-6)}{2} = 3$$

$x=0$: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{n\,3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n}$

Now $\lim_{n\to\infty}\dfrac{1}{n} = 0$ *and* $\dfrac{1}{n}$ *is a decreasing sequence. Therefore, this series is convergent by the Alternating Series Test.*

$x=-6$: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(-6+3)^n}{n\,3^n} = \sum_{n=0}^{\infty}\frac{1}{n}$

This series diverges by the p*-test.*

Center: $x=-3$
Radius of Convergence: $R=3$
Interval of Convergence: $(-6,0]$