Name: *Caleb McWhorter — Solutions* **Quiz 10: Series Tests**

MAT 296 **Fall 2017**

Problem 1: Determine whether the following series converges or diverges. If it converges, find the sum. If it diverges, prove it.

$$\sum_{n=1}^{\infty} \frac{5n^2+1}{4n^2-1}$$

$$\lim_{n\to\infty} \frac{5n^2+1}{4n^2-1} = \frac{5}{4} \neq 0$$

Therefore, $\sum_{n=1}^{\infty} \frac{5n^2+1}{4n^2-1}$ *diverges by the Divergence Test.*

Problem 2: Determine whether the following series converges or diverges. If it converges, find the sum. If it diverges, prove it.

$$\sum_{n=2}^{\infty} \frac{3}{n^2+n-2}$$

$$\frac{3}{n^2+n-2} = \frac{3}{(n-1)(n+2)} = \frac{A}{n-1} + \frac{B}{n+2}$$

Heaviside's Method gives $A = \frac{3}{1+2} = 1$ *and* $B = \frac{3}{-2-1} = -1$.

$$\sum_{n=2}^{\infty} \frac{3}{n^2+n-2} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+2} \right) = \left(\frac{1}{1} - \cancel{\frac{1}{4}} \right) + \left(\frac{1}{2} - \cancel{\frac{1}{5}} \right) + \left(\frac{1}{3} - \cancel{\frac{1}{4}} \right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} \right) + \cdots$$

So for sufficiently large N*, the* N*th partial sum is*

$$S_N = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} = \frac{11}{6} - \frac{1}{N+1}$$

Therefore, we have

$$\sum_{n=2}^{\infty} \frac{3}{n^2+n-2} := \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{11}{6} - \frac{1}{n+1} \right) = \frac{11}{6}$$

Problem 3: Determine whether the following series converges or diverges. Prove your answer.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

First, observe that $\dfrac{\ln x}{x^2} > 0$ *for* $x \geq 1$. *Furthermore,*

$$\frac{d}{dx}\left(\frac{\ln x}{x^2}\right) = \frac{x^2 \cdot \frac{1}{x} - 2x \ln x}{x^4} = \frac{x(1 - 2\ln x)}{x^4}$$

Notice $x^4 > 0, x > 0$ *for* $x \geq 1$. *Now if* $x \geq \sqrt{e}$, *then* $\ln x \geq \frac{1}{2}$ *so that* $2\ln x \geq 1$ *and then* $1 - 2\ln x \leq 0$. *Clearly,* $1 < \sqrt{e} < 2$. *But the series* $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ *converges if and only if* $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ *converges. From the work above, the Integral Test applies to this latter series.*

$\ln x$	$-\frac{1}{x}$
$\frac{1}{x}$	$\frac{1}{x^2}$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \int -\frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx := \lim_{b\to\infty} \int_2^{b} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Bigg|_2^b = \lim_{b\to\infty} \left(-\frac{\ln b}{b} - \frac{1}{b}\right) - \left(-\frac{\ln 2}{2} - \frac{1}{2}\right) = \frac{1 + \ln 2}{2}$$

Since the integral converges, by the Integral Test the series $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ *converges. But then so to must the series* $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ *converge.*