Name: *Caleb McWhorter — Solutions* **Quiz 14: Taylor Series**
MAT 296 **Fall 2017**

Problem 1: Find a Taylor Series for the function $f(x) = xe^{3x^2}$ about $x = 0$.

We know the Taylor Series about $x = 0$ for e^x: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Then we have

$$e^{3x^2} = \sum_{n=0}^{\infty} \frac{(3x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!}$$

Therefore,

$$f(x) = xe^{3x^2} = x \cdot \sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^{2n+1}}{n!}$$

Problem 2: Find a Taylor Series for the function $f(x) = \frac{x}{(1-x)^2}$ about $x = 0$. Use this to find the value of $\sum_{n=0}^{\infty} \frac{n}{2^n}$.

We know the Taylor Series for $\frac{1}{1-x}$ about $x = 0$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

But observe

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \sum_{n=0}^{\infty} nx^{n-1}$$

Therefore,

$$f(x) = \frac{x}{(1-x)^2} = x \cdot \frac{1}{(1-x)^2} = x \cdot \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} nx^n$$

This is valid for $|x| < 1$. But then we have

$$\sum_{n=0}^{\infty} \frac{n}{2^n} = \sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1/2}{(1-1/2)^2} = 2$$

Problem 3: Find a power series representation for

$$\int \cos x^2 \, dx$$

We know that a Taylor Series for $\cos x$ *about* $x = 0$ *is* $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. *Therefore, we have*

$$\cos x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

Thus,

$$\begin{aligned}
\int \cos x^2 \, dx &= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} \, dx \\
&= \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{4n}}{(2n)!} \, dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^{4n} \, dx \\
&= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{x^{4n+1}}{4n+1} \\
&= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}
\end{aligned}$$