Name: *Caleb McWhorter — Solutions* **Quiz 3: Trigonometric Integrals**
MAT 296 **Fall 2017**

Problem 1: Integrate the following:

$$\int \sin^2\theta\; d\theta$$

$$\int \sin^2\theta\; d\theta = \int \left(\frac{1-\cos 2\theta}{2}\right) d\theta = \frac{1}{2}\int (1-\cos 2\theta)\, d\theta = \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

Problem 2: Integrate the following:

$$\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx$$

$$\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx = \int \frac{\tan^2 x}{\sqrt{\sec x}} \cdot \tan x\; dx = \int \frac{\sec^2 - 1}{\sqrt{\sec x}} \cdot \tan x\; dx$$

$$u = \sec x$$

$$du = \sec x \tan x\; dx \iff du = u \tan x\; dx \iff \frac{du}{u} = \tan x\; dx$$

$$\int \frac{\sec^2 - 1}{\sqrt{\sec x}} \cdot \tan x\; dx = \int \frac{u^2-1}{\sqrt{u}}\frac{du}{u} = \int \frac{u^2-1}{u^{3/2}}\, du = \int \left(u^{1/2} - u^{-3/2}\right) du = \frac{2}{3}u^{3/2} + 2u^{-1/2} + C$$

Therefore, we have

$$\begin{aligned} \int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx &= \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C \\ &= 2\left(\frac{(\sec x)^{3/2}}{3} \cdot \frac{\sqrt{\sec x}}{\sqrt{\sec x}} + \frac{1}{\sqrt{\sec x}} \cdot \frac{3}{3}\right) + C \\ &= 2\, \frac{\sec^2 x + 3}{3\sqrt{\sec x}} + C \end{aligned}$$

Problem 3: Integrate the following:

$$\int \sin^8(\pi x)\cos^3(\pi x)\,dx$$

Let $u = \pi x$*. Then* $du = \pi\,dx$ *so that* $dx = \frac{du}{\pi}$.

$$\begin{aligned}\int \sin^8(\pi x)\cos^3(\pi x)\,dx &= \frac{1}{\pi}\int \sin^8 u\cos^3 u\,du\\ &= \frac{1}{\pi}\int \sin^8 u\cos^2 u\cdot\cos u\,du\\ &= \frac{1}{\pi}\int \sin^8 u\,(1-\sin^2 u)\cdot\cos u\,du\end{aligned}$$

Let $v = \sin u$ *so that* $dv = \cos u\,du$.

$$\begin{aligned}\frac{1}{\pi}\int \sin^8 u\,(1-\sin^2 u)\cdot\cos u\,du = \frac{1}{\pi}\int v^8(1-v^2)\,dv = \frac{1}{\pi}\int \left(v^8 - v^{10}\right)dv &= \frac{1}{\pi}\left(\frac{v^9}{9}-\frac{v^{11}}{11}\right)+C\\ &= \frac{1}{\pi}\left(\frac{\sin^9 u}{9}-\frac{\sin^{11} u}{11}\right)+C\\ &= \frac{1}{\pi}\left(\frac{\sin^9(\pi x)}{9}-\frac{\sin^{11}(\pi x)}{11}\right)+C\end{aligned}$$

Problem 4: Integrate the following:

$$\int \cot^3\theta\csc^5\theta\,d\theta$$

Recall $\sin^2\theta+\cos^2\theta = 1$*. Dividing by* $\sin^2\theta$ *yields* $1+\cot^2\theta = \csc^2\theta$ *so that* $\cot^2\theta = \csc^2\theta - 1$*. Now we have*

$$\int \cot^3\theta\csc^5\theta\,d\theta = \int \cot^2\theta\csc^4\theta\cdot\cot\theta\csc\theta\,d\theta = \int (\csc^2\theta - 1)\csc^4\theta\cdot\cot\theta\csc\theta\,d\theta$$

Let $u = \csc\theta$ *so that* $du = -\csc\theta\cot\theta\,d\theta$.

$$\begin{aligned}\int (\csc^2\theta-1)\csc^4\theta\cdot\cot\theta\csc\theta\,d\theta &= \int (u^2-1)\,u^4\cdot(-1)\,du\\ &= \int (1-u^2)u^4\,du\\ &= \int (u^4-u^6)\,du\\ &= \frac{u^5}{5}-\frac{u^7}{7}+C\\ &= \frac{\csc^5\theta}{5}-\frac{\csc^7\theta}{7}+C\end{aligned}$$