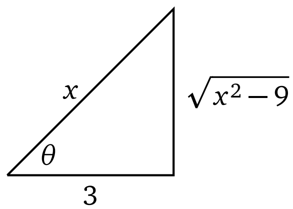
Name: *Caleb McWhorter — Solutions* **Quiz 5: Trig Sub**
MAT 296 **Fall 2017**

Problem 1: Integrate the following:

$$\int \frac{x^3}{\sqrt{x^2-9}}\,dx$$



$$\begin{aligned} a^2+b^2 &= c^2 \\ b^2 &= c^2-a^2 \\ &= x^2-9 \end{aligned}$$

$$\begin{aligned} \sec\theta &= \frac{x}{3} \\ x &= 3\sec\theta \\ dx &= 3\sec\theta\tan\theta\;d\theta \\ \hline x^3 &= 3^3\sec^3\theta \\ \tan\theta &= \frac{\sqrt{x^2-9}}{3} \\ \sqrt{x^2-9} &= 3\tan\theta \end{aligned}$$

$$\int \frac{x^3}{\sqrt{x^2-9}}\,dx = \int \frac{3^3\sec^3\theta}{3\tan\theta}\cdot 3\sec\theta\tan\theta\;d\theta = 3^3\int \sec^4\theta\;d\theta = 3^3\int \sec^2\theta\cdot\sec^2\theta\;d\theta = 3^3\int (1+\tan^2\theta)\cdot\sec^2\theta\;d\theta$$

Let $u=\tan\theta$, then $du=\sec^2\theta\;d\theta$.

$$3^3\int (1+\tan^2\theta)\cdot\sec^2\theta\;d\theta = 27\int (1+u^2)\,du = 3^3\left(u+\frac{u^3}{3}\right)+C = 3^3\left(\tan\theta+\frac{\tan^3\theta}{3}\right)+C$$

But $\tan\theta = \dfrac{(x^2-9)^{1/2}}{3}$ so that

$$\int \frac{x^3}{\sqrt{x^2-9}}\,dx = 27\left(\tan\theta+\frac{\tan^3\theta}{3}\right)+C = 3^3\left(\frac{\sqrt{x^2-9}}{3}+\frac{\sqrt{(x^2-9)^3}}{3^4}\right)+C = 9\sqrt{x^2-9}+\frac{\sqrt{(x^2-9)^3}}{3}+C$$

Problem 2: Integrate the following:

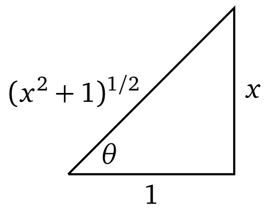
$$\int \frac{x}{\sqrt{x^2+9}}\,dx$$

Let $u=x^2+9$ *so that* $du=2x\;dx \Longleftrightarrow dx=\dfrac{du}{2x}$.

$$\int \frac{x}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u}+C = \sqrt{x^2+9}+C$$

Problem 3: Integrate the following:

$$\int_0^1 \frac{dx}{(x^2+1)^{5/2}}$$



$$a^2+b^2=c^2$$
$$x^2+1=c^2$$

$$\tan\theta = \frac{x}{1}$$
$$x = \tan\theta$$
$$dx = \sec^2\theta\; d\theta$$

Notice that

$$\cos\theta = \frac{1}{(x^2+1)^{1/2}}$$
$$(x^2+1)^{1/2} = \sec\theta$$
$$(x^2+1)^{5/2} = \sec^5\theta$$

$x=\tan\theta$ *so when* $x=0$, $\theta=0$ *and when* $x=1$, $\theta=\frac{\pi}{4}$.

$$\int_0^{\pi/4} \frac{\sec^2\theta}{\sec^5\theta}\, d\theta = \int_0^{\pi/4} \cos^3\theta\; d\theta = \int_0^{\pi/4} \cos^2\theta\cdot\cos\theta\; d\theta = \int_0^{\pi/4} (1-\sin^2\theta)\cdot\cos\theta\; d\theta$$

Let $u=\sin\theta$ *so that* $du=\cos\theta\; d\theta$. *When* $\theta=0$, $u=0$ *and if* $\theta=\pi/4$, $u=1/\sqrt{2}$.

$$\int_0^{1/\sqrt{2}} (1-u^2)\, du = \left(u-\frac{u^3}{3}\right)\Bigg|_0^{1/\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}} = \frac{5}{6\sqrt{2}}$$

Problem 4: Integrate the following:

$$\int \frac{dx}{x^2-6x+14}$$

$$\int \frac{dx}{x^2-6x+14} = \int \frac{dx}{x^2-6x+9-9+14} = \int \frac{dx}{(x-3)^2+5}\cdot\frac{1/5}{1/5} = \frac{1}{5}\int \frac{dx}{\left(\frac{x-3}{\sqrt{5}}\right)^2+1}$$

Let $u=\frac{x-3}{\sqrt{5}}$ *so that* $du=\frac{1}{\sqrt{5}}\,dx \Longleftrightarrow dx=\sqrt{5}\,du$.

$$\frac{1}{5}\int \frac{dx}{\left(\frac{x-3}{\sqrt{5}}\right)^2+1} = \frac{\sqrt{5}}{5}\int\frac{du}{u^2+1} = \frac{\sqrt{5}}{5}\arctan u + C = \frac{1}{\sqrt{5}}\arctan u + C = \frac{1}{\sqrt{5}}\arctan\left(\frac{x-3}{\sqrt{5}}\right) + C$$