

Problem 1: Integrate the following:

$$\int_0^\infty \frac{2x}{x^4+1} dx$$

$$\int \frac{2x}{x^4+1} dx = \int \frac{2x}{(x^2)^2+1} dx$$

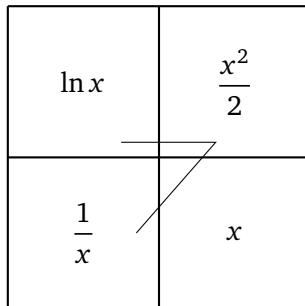
Let $u = x^2$ then $du = 2x dx$. Then

$$\int \frac{2x}{(x^2)^2+1} dx = \int \frac{du}{u^2+1} = \arctan u + C = \arctan(x^2) + C$$

$$\int_0^\infty \frac{2x}{x^4+1} dx := \lim_{b \rightarrow \infty} \int_0^b \frac{2x}{x^4+1} dx = \lim_{b \rightarrow \infty} \arctan(x^2) \Big|_0^b = \lim_{b \rightarrow \infty} \arctan(b^2) - \arctan 0 = \frac{\pi}{2}$$

Problem 2: Integrate the following:

$$\int_0^1 x \ln x dx$$



$$\int x \ln x dx = \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx = \frac{1}{2} x^2 \ln x - \frac{x^2}{4} + C$$

Note that using l'Hôpital's Rule:

$$\lim_{x \rightarrow 0^+} x^2 \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x^2} \stackrel{L.H.}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-2/x^3} = \lim_{x \rightarrow 0^+} \frac{x^3}{-2x} = \lim_{x \rightarrow 0^+} \frac{x^2}{-2} = 0$$

$$\int_0^1 x \ln x dx := \lim_{b \rightarrow 0^+} \int_b^1 x \ln x dx = \lim_{b \rightarrow 0^+} \left(\frac{1}{2} x^2 \ln x - \frac{x^2}{4} \right) \Big|_b^1 = \left(0 - \frac{1}{4} \right) - \lim_{b \rightarrow 0^+} \left(\frac{1}{2} b^2 \ln b - \frac{b^2}{4} \right) = -\frac{1}{4} - 0 = -\frac{1}{4}$$

Problem 3: Integrate the following:

$$\int_{-\infty}^{\infty} xe^{-x^2} dx$$

Let $u = x^2$ so that $du = 2x dx$ and $dx = \frac{du}{2x}$. Then

$$\int xe^{-x^2} dx = \frac{1}{2} \int e^{-u} du = -\frac{1}{2}e^{-u} + C = -\frac{e^{-x^2}}{2} + C$$

Then this gives

$$\begin{aligned} \int_{-\infty}^{\infty} xe^{-x^2} dx &:= \int_{-\infty}^0 xe^{-x^2} dx + \int_0^{\infty} xe^{-x^2} dx \\ &:= \lim_{b \rightarrow \infty} \int_{-b}^0 xe^{-x^2} dx + \lim_{b \rightarrow \infty} \int_0^b xe^{-x^2} dx \\ &= \lim_{b \rightarrow \infty} -\frac{e^{-x^2}}{2} \Big|_{-b}^0 + \lim_{b \rightarrow \infty} -\frac{e^{-x^2}}{2} \Big|_0^b \\ &= \left(-\frac{1}{2} - \lim_{b \rightarrow \infty} \frac{-e^{-b^2}}{2} \right) + \left(\lim_{b \rightarrow \infty} \frac{-e^{-b^2}}{2} - \frac{-1}{2} \right) \\ &= \left(\frac{-1}{2} - 0 \right) + \left(0 - \frac{-1}{2} \right) \\ &= -\frac{1}{2} + \frac{1}{2} \\ &= 0 \end{aligned}$$

Note: One should expect this as the integrand is an odd function. But we only know for sure that the integral over $[-a, a]$ of an odd function is 0 and not necessarily over the interval $(-\infty, \infty)$. Think of the case of $\int_{-\infty}^{\infty} x dx$, which surely does not exist.