Name: *Caleb McWhorter — Solutions* **Quiz 8: Improper Integrals**
MAT 296 **Fall 2017**

Problem 1: Integrate the following:

$$\int_0^\infty \frac{2x}{x^4+1}\,dx$$

$$\int \frac{2x}{x^4+1}\,dx = \int \frac{2x}{(x^2)^2+1}\,dx$$

Let $u = x^2$ then $du = 2x\,dx$. Then

$$\int \frac{2x}{(x^2)^2+1}\,dx = \int \frac{du}{u^2+1} = \arctan u + C = \arctan(x^2) + C$$

$$\int_0^\infty \frac{2x}{x^4+1}\,dx := \lim_{b\to\infty} \int_0^b \frac{2x}{x^4+1}\,dx = \lim_{b\to\infty} \arctan(x^2)\Big|_0^b = \lim_{b\to\infty} \arctan(b^2) - \arctan 0 = \frac{\pi}{2}$$

Problem 2: Integrate the following:

$$\int_0^1 x\ln x\,dx$$

$\ln x$	$\dfrac{x^2}{2}$
$\dfrac{1}{x}$	x

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{x^2}{4} + C$$

Note that using l'Hôpitals Rule:

$$\lim_{x\to 0^+} x^2\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{L.H.}{=} \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} \frac{x^3}{-2x} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0$$

$$\int_0^1 x\ln x\,dx := \lim_{b\to 0^+} \int_b^1 x\ln x\,dx = \lim_{b\to 0^+} \left(\frac{1}{2}x^2\ln x - \frac{x^2}{4}\right)\Bigg|_b^1 = \left(0 - \frac{1}{4}\right) - \lim_{b\to 0^+} \left(\frac{1}{2}b^2\ln b - \frac{b^2}{4}\right) = -\frac{1}{4} - 0 = -\frac{1}{4}$$

Problem 3: Integrate the following:

$$\int_{-\infty}^{\infty} xe^{-x^2}\, dx$$

Let $u = x^2$ *so that* $du = 2x\, dx$ *and* $dx = \dfrac{du}{2x}$. *Then*

$$\int xe^{-x^2}\, dx = \frac{1}{2}\int e^{-u}\, du = -\frac{1}{2}e^{-u} + C = -\frac{e^{-x^2}}{2} + C$$

Then this gives

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{-x^2}\, dx &:= \int_{-\infty}^{0} xe^{-x^2}\, dx + \int_{0}^{\infty} xe^{-x^2}\, dx \\
&:= \lim_{b\to\infty}\int_{-b}^{0} xe^{-x^2}\, dx + \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}\, dx \\
&= \lim_{b\to\infty} -\frac{e^{-x^2}}{2}\Bigg|_{-b}^{0} + \lim_{b\to\infty} -\frac{e^{-x^2}}{2}\Bigg|_{0}^{b} \\
&= \left(-\frac{1}{2} - \lim_{b\to\infty}\frac{-e^{-b^2}}{2}\right) + \left(\lim_{b\to\infty}\frac{-e^{-b^2}}{2} - \frac{-1}{2}\right) \\
&= \left(\frac{-1}{2} - 0\right) + \left(0 - \frac{-1}{2}\right) \\
&= -\frac{1}{2} + \frac{1}{2} \\
&= 0
\end{aligned}$$

Note: One should expect this as the integrand is an odd function. But we only know for sure that the integral over $[-a, a]$ *of an odd function is 0 and not necessarily over the interval* (∞, ∞). *Think of the case of* $\int_{-\infty}^{\infty} x\, dx$, *which surely does not exist.*