Math 397: Exam 1
Summer Session II – 2017
07/17/2017
90 Minutes

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Write your name on the appropriate line on the exam cover sheet. This exam contains 11 pages (including this cover page) and 9 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	9	
2	8	
3	7	
4	6	
5	9	
6	7	
7	7	
8	10	
9	9	
Total:	72	

1. (9 points) Answer the following questions:

(a) If $\mathbf{a} = \langle 2, -1, 0\rangle$ and $\mathbf{b} = \langle 5, -3, 4\rangle$, find $|\mathbf{a}|$ and $2\mathbf{a} - \mathbf{b}$.

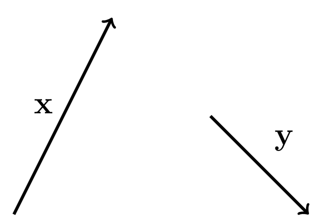
$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{4+1+0} = \sqrt{5}$$

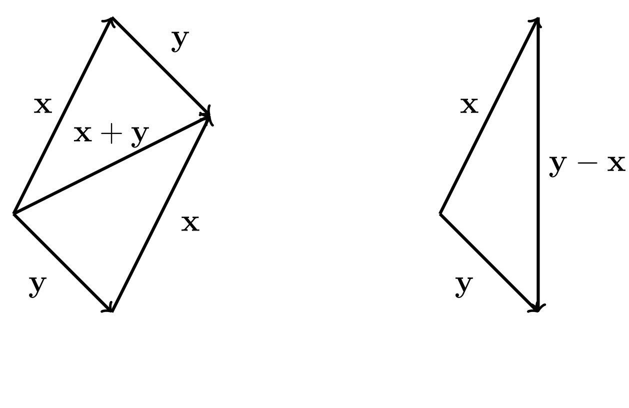
$$2\vec{a} - \vec{b} = 2\langle 2, -1, 0\rangle - \langle 5, -3, 4\rangle = \langle 4, -2, 0\rangle - \langle 5, -3, 4\rangle = \langle -1, 1, -4\rangle$$

(b) Find a unit vector in the direction of the vector $\langle 3, 1\rangle$.

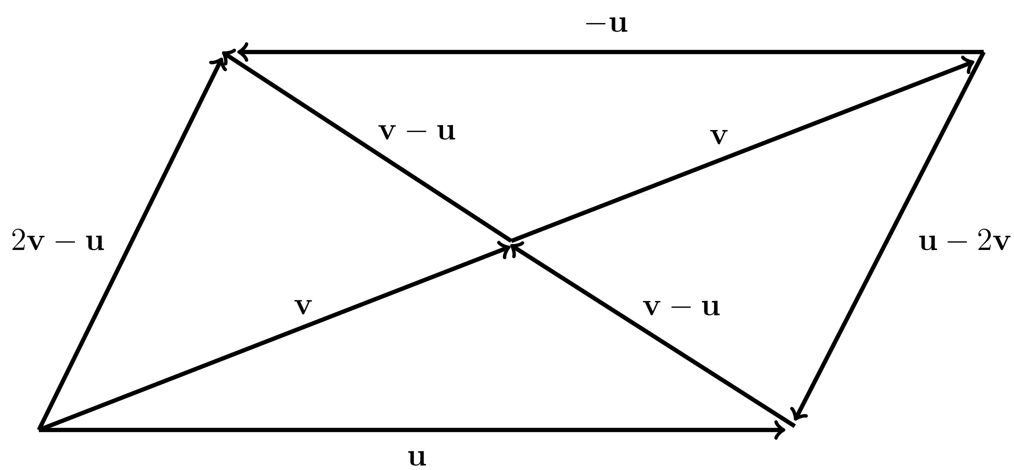
$$\frac{\langle 3,1\rangle}{|\langle 3,1\rangle|} = \frac{\langle 3,1\rangle}{\sqrt{3^2+1^2}} = \frac{\langle 3,1\rangle}{\sqrt{10}} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$$

(c) Given the vectors $\mathbf{x}$ and $\mathbf{y}$ shown below, find $\mathbf{x} + \mathbf{y}$ and $\mathbf{y} - \mathbf{x}$.

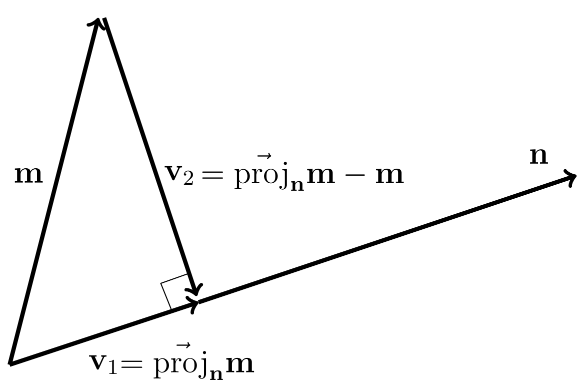




(d) In the vector diagram below, label all vectors in terms of the given vectors $\mathbf{u}$ and $\mathbf{v}$.



(e) Given the vector diagram below, find $\mathbf{v}_1$ and $\mathbf{v}_2$ in terms of $\mathbf{m}$ and $\mathbf{n}$.



2. (8 points) Complete the following parts:

(a) Let $\mathbf{u} = 2\,\hat{\imath} + 3\,\hat{\mathbf{k}}$ and $\mathbf{v} = 5\,\hat{\imath} + \hat{\jmath} - 2\,\hat{\mathbf{k}}$, find $\mathbf{u} \cdot \mathbf{v}$.

$$\mathbf{u} \cdot \mathbf{v} = (2\,\hat{\imath} + 3\,\hat{\mathbf{k}}) \cdot (5\,\hat{\imath} + \hat{\jmath} - 2\,\hat{\mathbf{k}}) = 2(5) + 0(1) + 3(-2) = 10 + 0 - 6 = 4$$

(b) Find a nonzero vector perpendicular to $5\,\hat{\imath} + 2\,\hat{\jmath} - \hat{\mathbf{k}}$.

If $\langle x, y, z\rangle$ is perpendicular to $5\,\hat{\imath} + 2\,\hat{\jmath} - \hat{\mathbf{k}}$, then their dot product must give 0. But then $0 = \langle x, y, z\rangle \cdot (5\,\hat{\imath} + 2\,\hat{\jmath} - \hat{\mathbf{k}}) = 5x + 2y - z$. Then $\pm\langle 2, -5, 0\rangle$, , $\pm\langle 1, 0, 5\rangle$, $\pm\langle 0, 1, 2\rangle$, and $\pm\langle 1, -1, 3\rangle$ and their scalar multiples are examples.

(c) The curves $y = x^2 - 1$ and $y = \dfrac{5}{2}x - 2$ intersect at the point $(2, 3)$, find the acute angle between the curves at this point of intersection.

Let $f(x) = x^2 - 1$ and $g(x) = \dfrac{5}{2}x - 2$. We know f, g intersect at $(2, 3)$. We know also

$$\begin{aligned} f'(x) &= 2x\Big|_{x=2} = 4 \\ g'(x) &= \frac{5}{2}\Big|_{x=2} = \frac{5}{2} \end{aligned}$$

Which gives direction vectors $\langle 1, 4\rangle$ and $\langle 2, 5\rangle$ for f, g at $(2, 3)$, respectively. Then we have

$$\begin{aligned} \langle 1, 4\rangle \cdot \langle 2, 5\rangle &= |\langle 1, 4\rangle|\,|\langle 2, 5\rangle|\,\cos\theta \\ 22 &= \sqrt{17} \cdot \sqrt{29} \cdot \cos\theta \\ \frac{22}{\sqrt{493}} &= \cos\theta \end{aligned}$$

so that $\theta = \arccos(22/\sqrt{493}) \approx 7.765°$

3. (7 points) Complete the following parts:

(a) Find a vector perpendicular to both $\mathbf{u} = \langle 0, -1, 1\rangle$ and $\mathbf{v} = \langle 5, -3, -1\rangle$.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{k}} & \hat{\mathbf{k}} \\ 0 & -1 & 1 \\ 5 & 03 & -1 \end{vmatrix} = \hat{\mathbf{i}}\begin{vmatrix} -1 & 1 \\ -3 & -1 \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} 0 & 1 \\ 5 & -1 \end{vmatrix} + \hat{\mathbf{k}}\begin{vmatrix} 0 & -1 \\ 5 & -3 \end{vmatrix}$$
$$= (1 - -3)\hat{\mathbf{i}} - \hat{\mathbf{j}}(0-5) + \hat{\mathbf{k}}(0 - -5)$$
$$= \langle 4, 5, 5\rangle$$

so that $\langle 4k, 5k, 5k\rangle$ *will work, where* $k \neq 0$*. In particular choosing* $k = 1$*,* $\langle 4, 5, 5\rangle$ *is such a vector.*

(b) Find the area of the triangle with vertices $P(1, 0, 1)$, $Q(-1, 1, 1)$, and $R(0, 2, 1)$.

Let $\vec{QP} = \langle 1, 0, 1\rangle - \langle -1, 1, 1\rangle = \langle 2, -1, 0\rangle$ *and* $\vec{QR} = \langle 0, 2, 1\rangle - \langle -1, 1, 1\rangle = \langle 1, 1, 0\rangle$*. Then we have*

$$\vec{QP}\times\vec{QR} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \hat{\mathbf{i}}(0) - \hat{\mathbf{j}}(0) + \hat{\mathbf{k}}(2 - -1) = 3\hat{\mathbf{k}}$$

The area is half the area of the spanned parallelogram, which is $|3\hat{\mathbf{k}}| = 3$*. So the area of the triangle is* $A = 3/2$*.*

(c) Find the volume of the parallelepiped formed by $\mathbf{u} = \langle 2, 1, -1\rangle$, $\mathbf{v} = \langle 1, 2, 4\rangle$, and $\mathbf{w} = \langle 2, 2, 2\rangle$. What, if anything, does this answer imply about $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$?

The volume is given by the scalar triple product, $V = |\vec{a}\cdot\vec{b}\times\vec{c}|$*. We have*

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 4 \\ 2 & 2 & 2 \end{vmatrix} = \hat{\mathbf{i}}(4-8) - \hat{\mathbf{j}}(2-8) + \hat{\mathbf{k}}(2-4) = \langle -4, 6, -2\rangle$$

and then $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \langle 2, 1, -1\rangle\cdot\langle -4, 6, -2\rangle = -8 + 6 + 2 = 0$*. Alternatively, we can compute*

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 4 \\ 2 & 2 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$$
$$= 2(4-8) - (2-8) - (2-4) = 2(-4) - (-6) - (-2) = -8 + 6 + 2 = 0$$

The only way for the volume of the parallelepiped to be 0 is for the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *to be coplanar.*

4. (6 points) Find a parametrization for the following curves:

(a) The line segment from the point $(7, -1, 4)$ to the point $(-3, 2, 3)$.

$$\begin{aligned}\ell(t) &= (-3, 2, 3)t + (7, -1, 4)(1 - t)\\ &= (-3t, 2t, 3t) + (7 - 7t, t - 1, 4 - 4t)\\ &= (7 - 10t, 3t - 1, 4 - t)\end{aligned}$$

for $0 \leq t \leq 1$ is a possible parametrization.

(b) The ellipse, oriented counterclockwise, given by

$$\frac{(x-2)^2}{3} + \frac{(y+1)^2}{5} = 1$$

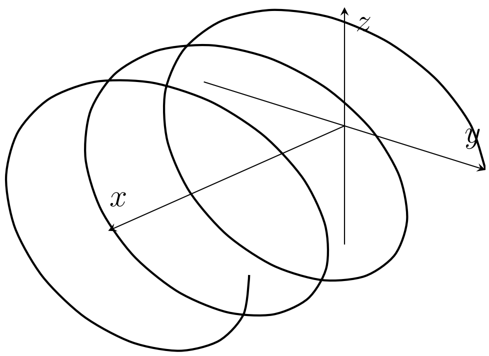
This is an ellipse centered at $(2, -1)$ with semimajor axis 'radius' $\sqrt{5}$ in the y-direction and semiminor axis 'radius' $\sqrt{3}$ in the x direction. Then we have $(\sqrt{3}\cos t + 2, \sqrt{5}\sin t - 1)$ for $0 \leq t \leq 2\pi$ as a possible parametrization.

(c) The curve given by $f(x) = x\cos x + e^x \sin^2 \sqrt[3]{x}$.

We use $(t, f(t))$ so that $(t, t\cos t + e^t \sin^2 \sqrt[3]{t})$ is a possible parametrization.

5. (9 points) Suppose the path of a photon emitted from a hydrogen atom is given by $\mathbf{x}(t) = (t, \cos t, \sin t)$.

   (a) Sketch the path of the particle. Find the position of the particle at $t = \pi$.



$$\mathbf{x}(\pi) = (\pi, \cos\pi, \sin\pi) = (\pi, -1, 0).$$

   (b) Find the velocity of the particle at $t = \frac{\pi}{4}$.

$$\mathbf{x}'(t) = (1, -\sin t, \cos t)\Big|_{t=\frac{\pi}{4}} = \left(1, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

   (c) If the electron orbiting the atom has acceleration vector given by $\mathbf{a}(t) = 2\,\hat{\imath} - \cos(1-t)\,\hat{\mathbf{k}}$ and the electron was located at $2\hat{\imath} + 3\hat{\jmath} + \hat{\mathbf{k}}$ with velocity vector $2\hat{\imath}$ at $t = 1$, give a vector valued function for the electron's position.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 2, 0, -\cos(1-t)\rangle\,dt = \langle 2t, 0, \sin(1-t)\rangle + \langle C_1, C_2, C_3\rangle$$
$$\mathbf{v}(1) = \langle 2, 0, 0\rangle = \langle 2, 0, 0\rangle + \langle C_1, C_2, C_3\rangle$$

so that $C_1 = 0, C_2 = 0, C_3 = 0$. *Then* $\mathbf{v}(t) = \langle 2t, 0, \sin(1-t)\rangle$. *Then*

$$\mathbf{p}(t) = \int \mathbf{v}(t)\,dt = \int \langle 2t, 0, \sin(1-t)\rangle\,dt = \langle t^2, 0, \cos(1-t)\rangle + \langle D_1, D_2, D_3\rangle$$
$$\mathbf{p}(1) = \langle 2, 3, 1\rangle = \langle 1, 0, 1\rangle + \langle D_1, D_2, D_3\rangle$$

so that $D_1 = 1, D_2 = 3, D_3 = 0$. *Therefore,* $p(t) = \langle t^2 + 1, 3, \cos(1-t)\rangle$.

6. (7 points) Complete the following parts:

(a) Find the (vector) parametric equation of the line containing the point $(0, 5, 3)$ and parallel to the vector $\langle 2, -3, 1 \rangle$.

$$\ell(t) = (0, 5, 3) + t(2, -3, 1) = (2t, 5 - 3t, t + 3)$$

Of course, either of the following are fine:

$$\begin{cases} x = 2t \\ y = 5 - 3t \\ z = t + 3 \end{cases} \qquad \textit{OR} \qquad \frac{x - 0}{2} = \frac{y - 5}{-3} = \frac{z - 3}{1}$$

(b) Find the (vector) parametric equation of the line perpendicular to the plane $x - 3y = 13$ and containing the point $(-1, 0, 1)$.

$$\ell(t) = (-1, 0, 1) + t(1, -3, 0) = (t - 1, -3t, 1)$$

Of course, either of the following are fine:

$$\begin{cases} x = t - 1 \\ y = -3t \\ z = 1 \end{cases} \qquad \textit{OR} \qquad \frac{x + 1}{1} = \frac{y - 0}{-3}\ ;\ z = 1$$

(c) Determine if the lines $l_1(t) = (t, 1 - t, 1 - t)$ and $l_2(s) = (6s, -s - 1, 3s + 1)$ are the same, parallel, skew, or intersecting. If the lines intersect, find the point of intersection and determine if they are perpendicular.

ℓ_1 has direction vector $\langle 1, -1, -1 \rangle$ and ℓ_2 has direction vector $\langle 6, -1, 3 \rangle$. These are not parallel vectors so that ℓ_1 and ℓ_2 cannot be parallel lines or the same line. Setting the components equal, we fine

$$\begin{aligned} t &= 6s \\ 1 - t &= -s - 1 \\ 1 - t &= 3s + 1 \end{aligned}$$

Using the first equation, we find $t = 6s$. Using this in the second equation, we find $1 - 6s = -s - 1$ so that $2 = 5s$ so that $s = 2/5$. But then $t = 12/5$. Trying this solution pair in the last equation, we find $-7/5 = 11/5$, which is impossible. Therefore, ℓ_1 and ℓ_2 are skew lines (which automatically they neither intersect nor are perpendicular).

7. (7 points) Complete the following parts:

(a) Find a normal vector to the plane $5x + 3y + z = 15$ and find three points on the plane.

A possible normal vector is $\langle 5, 3, 1\rangle$. *Three 'easy' to find points are* $(3, 0, 0)$, $(0, 5, 0)$, *and* $(0, 0, 15)$. *Also,* $(2, 1, 2)$ *is on the plane. In fact,* $(s, t, 15 - 5s - 3t)$ *is in the plane for all* $s, t \in \mathbb{R}$.

(b) Find the equation of the plane with normal vector $\langle 2, 3, -1\rangle$ and containing the point $(1, 0, 2)$.

$$\begin{aligned} \langle 2, 3, -1\rangle \cdot \langle x - 1, y - 0, z - 2\rangle &= 0 \\ 2(x - 1) + 3(y - 0) - (z - 2) &= 0 \\ 2x - 2 + 3y - z + 2 &= 0 \\ 2x + 3y - z &= 0 \end{aligned}$$

(c) Find the equation of the plane parallel to the plane $2x - y = 5 - 3z$ and containing the line $l(t) = (-1, 10, 4)t + (2, 5, 2)$.

Recall that parallel planes have parallel normal vectors. The plane given by $2x - y = 5 - 3z \Leftrightarrow 2x - y + 3z = 5$ *has a possible normal vector* $\langle 2, -1, 3\rangle$. *But then*

$$\begin{aligned} \langle 2, -1, 3\rangle \cdot \langle x - 2, y - 5, z - 2\rangle &= 0 \\ 2(x - 2) - (y - 5) + 3(z - 2) &= 0 \\ 2x - 4 - y + 5 + 3z - 6 &= 0 \\ 2x - y + 3z &= 5 \end{aligned}$$

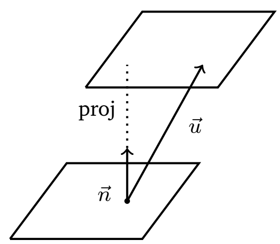
8. (10 points) Complete the following parts:

(a) Find the distance between the planes $2x - 5y + z = 4$ and $2x - 5y + z = 6$.

A possible normal vector to both is $\vec{n} = \langle 2, -5, 1\rangle$. Now $(0,0,4)$ is on the plane $2x - 5y + z = 4$ and $(0,0,6)$ is on the plane $2x - 5y + z = 6$. Then we have a vector $\vec{u} = \langle 0,0,6\rangle - \langle 0,0,4\rangle = \langle 0,0,2\rangle$. Then

$$\operatorname{proj}_{\vec{n}} \vec{u} = \frac{\langle 2,-5,1\rangle \cdot \langle 0,0,2\rangle}{\langle 2,-5,1\rangle \cdot \langle 2,-5,1\rangle} \langle 2,-5,1\rangle = \frac{0+0+2}{4+25+1} \langle 2,-5,1\rangle = \frac{2}{30}\langle 2,-5,1\rangle = \frac{1}{15}\langle 2,-5,1\rangle$$

But then the distance is $|\operatorname{proj}_{\vec{n}} \vec{u}| = |\frac{1}{15}\langle 2,-5,1\rangle| = \frac{1}{15}|\langle 2,-5,1\rangle| = \dfrac{\sqrt{30}}{15}$



(b) Find a parametrization for the intersection of $\Pi_1 : x+y-z = 4$ and $\Pi_2 : x-y+z = 6$.

The intersection of two planes Π_1 *and* Π_2 *is a line. The direction if the line is perpendicular to both direction vectors for the individual planes. So*

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \hat{\mathbf{i}}(1-1) - \hat{\mathbf{j}}(1 - -1) + \hat{\mathbf{k}}(-1-1) = \langle 0,-2,-2\rangle$$

is such a vector. We can use any parallel vector as direction for the line, a simple one being $-\frac{1}{2}\langle 0,-2,-2\rangle = \langle 0,1,1\rangle$. *We have a direction, we only need a point. We choose a random variable value and try to solve for the others: if* $z = 0$, *then* $x + y - 0 = 4$ *and* $x - y + 0 = 6$. *But this gives* $2x = 10$ *and* $x = 5$ *so that* $y = -1$. *Then the line of intersection is*

$$\ell(t) = (0,1,1)t + (5,-1,0) = (5, t-1, t)$$

Note that using $\langle 0,1,1\rangle$ *as the direction instead, we find* $(5, -2t-1, -2t)$.

9. (9 points)

(a) Identify the surfaces listed below:

(i) $x + y = 2z - 4$ — *plane*

(ii) $(x-1)^2 + y^2 + (z+2)^2 = 5$ — *sphere*

(iii) $z = 1 - 2x^2 - 3y^2$ — *paraboloid*

(iv) $y^2 - z^2 = x^2 + 1$ — *hyperboloid of two sheets*

(v) $x = y^2$ — *parabolic cylinder*

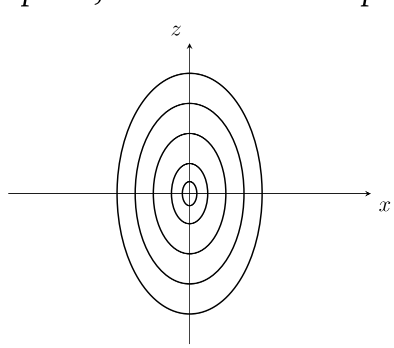
(vi) $y = \dfrac{z^2}{4} - x^2$ — *hyperbolic paraboloid*

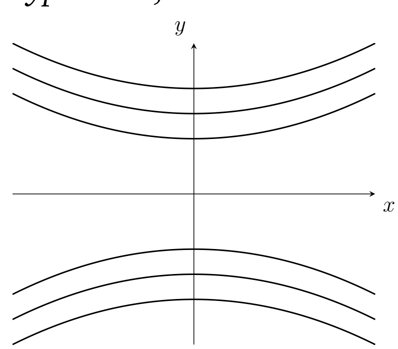
(vii) $y^2 + z^2 = 4$ — *cylinder*

(viii) $x^2 + y^2 - z^2 = 4$ — *hyperboloid of one sheet*

(b) Use appropriate level curves to sketch the surface $y^2 = x^2 + \dfrac{z^2}{4}$.

Note that slices in the y-plane are ellipses in the xz-plane, symmetric about the z-axis in the xz-plane, and slices in the z-plane are hyperbolas, as shown below.





Now putting these level curves together, we obtain the figure on the left below. Smoothing this figure gives the final surface shown below on the right.