Math 397: Exam 2 **Name:** *Caleb M^cWhorter — Solutions*
Summer Session II – 2017
07/27/2017
120 Minutes

Write your name on the appropriate line on the exam cover sheet. This exam contains 12 pages (including this cover page) and 8 questions. Check that you have every page of the exam. Answer the questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page — being sure to indicate the problem number.

Question	Points	Score
1	12	
2	10	
3	16	
4	10	
5	10	
6	15	
7	15	
8	12	
Total:	100	

1. (12 points) Evaluate the following limits. If the limit exists, prove that it exists; if the limit does not exist, explain why.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + y^2}{x^2 + y^2}$

Approaching along the x-axis, where $y = 0$:

$$\lim_{x\to 0} \frac{x^2 - 0 + 0^2}{x^2 + 0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Approaching along the line $x = y$:

$$\lim_{x\to 0} \frac{x^2 - x \cdot x + x^2}{x^2 + x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

(b) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{6xe^y - \sin y \cos x}{x^2 - y^2}$

Because $\dfrac{6xe^y - \sin y \cos x}{x^2 - y^2}$ *is continuous at* $(1, 0)$*,*

$$\lim_{(x,y)\to(1,0)} \frac{6xe^y - \sin y \cos x}{x^2 - y^2} = \frac{6(1)e^0 - \sin 0 \cos 1}{1^2 - 0^2} = \frac{6 - 0}{1} = 6$$

(c) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x^4 - 2x^3y - x^2y^2 + 2xy^3}{x^2 - 2xy}$

$$\begin{aligned}
\lim_{(x,y)\to(2,1)} \frac{x^4 - 2x^3y - x^2y^2 + 2xy^3}{x^2 - 2xy} &= \lim_{(x,y)\to(2,1)} \frac{x^3(x-2y) - xy^2(x-2y)}{x(x-2y)} \\
&= \lim_{(x,y)\to(2,1)} \frac{(x^3 - xy^2)(x-2y)}{x(x-2y)} \\
&= \lim_{(x,y)\to(2,1)} \frac{x(x-2y)(x^2-y^2)}{x(x-2y)} \\
&= \lim_{(x,y)\to(2,1)} (x^2 - y^2) \\
&= 2^2 - 1^2 = 3
\end{aligned}$$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{3x^2 + 2y^2}$

Observe that $x^2 \leq 3x^2 \leq 3x^2 + 2y^2$ *so that* $\dfrac{x^2}{3x^2+2y^2} \leq 1$. *But then*

$$\left|\frac{x^2 \sin^2 y}{3x^2 + 2y^2}\right| = \left|\sin^2 y \cdot \frac{x^2}{3x^2 + 2y^2}\right| \leq |\sin^2 y \cdot 1| = \sin^2 y$$

However, $\displaystyle\lim_{(x,y)\to(0,0)} \sin^2 y = 0$. *Therefore by the Squeeze Theorem,*

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{3x^2 + 2y^2} = 0$$

2. (10 points) Let $f(x,y,z) := \dfrac{x^y + y\ln(2y) - z}{\cos x}$. Find the following partial derivatives [you need not simplify]:

(a) $\dfrac{\partial f}{\partial x} = \dfrac{yx^{y-1}\cos x - (-\sin x)(x^y + y\ln(2y) - z)}{\cos^2 x}$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{x^y \ln x + \ln(2y) + 1}{\cos x}$

(c) $\dfrac{\partial f}{\partial z} = \dfrac{-1}{\cos x}$

(d) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\cos x(yx^{y-1}\ln x + x^{y-1}) - (-\sin x)(x^y \ln x + \ln(2y) + 1)}{\cos^2 x}$

(e) $\dfrac{\partial^2 f}{\partial z^2} = 0$

3. (16 points) Complete the following parts:

(a) Set up *but do not evaluate* an integral find the arclength of the curve given by $x(t) = \sqrt{2}\,t$, $y(t) = \dfrac{t^2}{2}$, and $z(t) = \ln t$ for $1 \le t \le 4$.

$$\begin{cases} x'(t) = \sqrt{2} \\ y'(t) = t \\ z'(t) = \dfrac{1}{t} \end{cases}$$

Therefore, we have

$$\int_1^4 \sqrt{(\sqrt{2})^2 + t^2 + (1/t)^2}\, dt$$

(b) Find the tangent plane to the function $f(x, y, z) = x^2y + 3z\cos(yz)$ at the point $(1, 2, 0)$.

$$\begin{aligned} \nabla f &= \langle 2xy, x^2 - 3z^3\sin(yz), 3\cos(yz) - 3yz\sin(yz)\rangle \\ \nabla f(1,2,0) &= \langle 2(1)2, 1^2 - 3(0)\sin 0, 3\cos 0 - 3(2)0\sin 0\rangle = \langle 4, 1, 3\rangle \end{aligned}$$

Then have

$$\begin{aligned} \langle 4, 1, 3\rangle \cdot \langle x-1, y-2, z-0\rangle &= 0 \\ 4(x-1) + 1(y-2) + 3(z-0) &= 0 \\ 4x - 4 + y - 2 + 3z &= 0 \\ 4x + y + 3z &= 6 \end{aligned}$$

(c) Find the total differential for the function $f(x, y, z) = \dfrac{y\sqrt[3]{x}}{z^2}$.

$$df = f_x\;dx + f_y\;dy + f_z\;dz$$
$$df = \frac{y}{3z^2\sqrt[3]{x^2}}\;dx + \frac{\sqrt[3]{x}}{z^2}\;dy + \frac{-2y\sqrt[3]{x}}{z^3}\;dz$$

(d) Use the method of total differentials to approximate $\sqrt[3]{7.4} \cdot \dfrac{0.9}{(1.1)^2}$.

We use the total differential of $f(x, y, z) = \dfrac{y\sqrt[3]{x}}{z^2}$ *at* $x = 8, y = 1, z = 1$. *We know that* $dz = -0.6 = \dfrac{-6}{10}$, $dy = -0.1 = \dfrac{-1}{10}$, $dz = 0.1 = \dfrac{1}{10}$. *Finally,* $f(8, 1, 1) = \dfrac{1 \cdot \sqrt[3]{8}}{1^2} = 2$ *so that*

$$\begin{aligned} df &= \frac{1}{3(1^2)\sqrt[3]{8^2}}\left(\frac{-6}{10}\right) + \frac{\sqrt[3]{8}}{1^2}\left(\frac{-1}{10}\right) + \frac{-2(1^3)\sqrt[3]{8}}{1^3}\left(\frac{1}{10}\right) \\ &= \frac{1}{12}\left(\frac{-6}{10}\right) + \frac{2}{1}\left(\frac{-1}{10}\right) - \frac{4}{1}\left(\frac{1}{10}\right) \\ &= \frac{-1}{20} - \frac{4}{20} - \frac{8}{20} \\ &= \frac{-13}{20} \end{aligned}$$

Then $f(7.4, 0.9, 1.1) \approx f(8, 1, 1) + df = 2 - \dfrac{13}{20} = \dfrac{40}{20} - \dfrac{13}{20} = \dfrac{27}{20} = 1.35$. *Note that the true value is* $1.44944\cdots$, *meaning we have approximated the true value with only a 6.9% error!*

4. (10 points) Complete the following parts:

(a) Let $f(x,y,z) = x^2z + yz^2 - \cos(xyz)$. Find $\nabla f(1,\pi,1)$.

$$\begin{aligned}
\nabla f &= \langle 2xz + yz\sin(xyz), z^2 + xz\sin(xyz), x^2 + 2yz + xy\sin(xyz)\rangle \\
\nabla f &= \langle 2(1)1 + \pi\sin\pi, 1^2 + \sin\pi, 1 + 2\pi\sin\pi\rangle \\
&= \langle 2+0, 1+0, 1+2\pi+0\rangle \\
&= \langle 2, 1, 1+2\pi\rangle
\end{aligned}$$

Let $\mathbf{V}(x,y,z) = (x+yz)\,\hat{\mathbf{\imath}} + (y+xz)\,\hat{\mathbf{\jmath}} + (z+xy)\,\hat{\mathbf{k}}$.

(b) Find $\mathrm{Div}\,\mathbf{V}$.

$$\begin{aligned}
\nabla\cdot\mathbf{V} &= \frac{\partial}{\partial x}(x+yz) + \frac{\partial}{\partial y}(y+xz) + \frac{\partial}{\partial z}(z+xy) \\
&= 1+1+1 \\
&= 3
\end{aligned}$$

(c) Find $\mathrm{curl}\,\mathbf{V}$.

$$\nabla\times\mathbf{V} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+yz & y+xz & z+xy \end{vmatrix}$$

$$= \hat{\mathbf{\imath}}\left(\frac{\partial}{\partial y}(z+xy) - \frac{\partial}{\partial z}(y+xz)\right) - \hat{\mathbf{\jmath}}\left(\frac{\partial}{\partial x}(z+xy) - \frac{\partial}{\partial z}(x+yz)\right) + \hat{\mathbf{k}}\left(\frac{\partial}{\partial x}(y+xz) - \frac{\partial}{\partial y}(x+yz)\right)$$

$$= \hat{\mathbf{\imath}}(x-x) - \hat{\mathbf{\jmath}}(y-y) + \hat{\mathbf{k}}(z-z)$$

$$= \langle 0,0,0\rangle$$

$$= \vec{\mathbf{0}}$$

5. (10 points) Let S be the surface given by $f(x,y) = x^2 - \dfrac{1}{y^2}$.

(a) Find $D_{\mathbf{u}} f(2,-1)$, where $\mathbf{u} = \langle 1, -1 \rangle$.

$$\begin{aligned}
\nabla f &= \left\langle 2x, \frac{2}{y^3} \right\rangle \\
\nabla f(2,-1) &= \langle 4, -2 \rangle \\
|\mathbf{u}| &= \sqrt{1^2 + (-1)^2} = \sqrt{2} \\
D_{\mathbf{u}} f(2,-1) &= \langle 4, -2 \rangle \cdot \frac{\langle 1, -1 \rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}(4(1) + (-2)(-1)) = \frac{6}{\sqrt{2}}
\end{aligned}$$

(b) Find the direction of steepest ascent and descent on this surface at $(2,-1)$.

The direction of steepest ascent is $\nabla f(2,-1) = \langle 4, -2 \rangle$*, while the direction of steepest descent is* $-\nabla f(2,-1) = \langle -4, 2 \rangle$.

(c) Find the maximum and minimum rates of change for $f(x,y)$ at the point $(2,-1)$.

The maximum rate of change is $|\langle 4, -2 \rangle| = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$*, while the minimum rate of change is* $-|\langle 4, -2 \rangle| = -\sqrt{20} = -2\sqrt{5}$.

6. (15 points) Identify and determine the nature of any critical points for the function $f(x, y, z) := x^3 + xz^2 - 3x^2 + y^2 + 2z^2$.

$$
\begin{aligned}
f_x &= 3x^2 + z^2 - 6x & f_{xx} &= 6x - 6 = 6(x-1) \\
f_y &= 2y & f_{yy} &= 2 \\
f_z &= 2xz + 4z = 2z(x+2) & f_{zz} &= 2(x+2) \\
& & f_{xy} &= 0 \\
& & f_{xz} &= 2z \\
& & f_{yz} &= 0
\end{aligned}
$$

Therefore, the critical values satisfy the equations

$$
\begin{cases}
3x^2 + z^2 - 6x & = 0 \\
2y & = 0 \\
2z(x+2) & = 0
\end{cases}
$$

Now $2y = 0$ *so that* $y = 0$. *But* $2z(x+2) = 0$ *implies that* $2z = 0$ *so that* $z = 0$ *or* $x + 2 = 0$ *so that* $x = -2$. *If* $z = 0$, *then*

$$
\begin{aligned}
3x^2 + z^2 - 6x &= 0 \\
3x^2 + 0 - 6x &= 0 \\
3x(x-2) &= 0
\end{aligned}
$$

so that $x = 0$ *or* $x = 2$. *This gives solutions* $(0, 0, 0)$ *and* $(2, 0, 0)$. *Now if* $x = -2$, *then*

$$
\begin{aligned}
3x^2 + z^2 - 6x &= 0 \\
3(-2)^2 + z^2 - 6(-2) &= 0 \\
z^2 + 14 &= 0
\end{aligned}
$$

which implies that $z^2 = -14$, *which is impossible. Therefore (using the fact that* $f(x, y, z)$ *is smooth), the only critical values are* $(0, 0, 0)$ *and* $(2, 0, 0)$. *The Jacobian is*

$$
\begin{pmatrix}
6(x-1) & 0 & 2z \\
0 & 2 & 0 \\
2z & 0 & 2(x+2)
\end{pmatrix}
$$

At $(0, 0, 0)$, *the matrix is*

$$
\begin{pmatrix}
-6 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 4
\end{pmatrix}
$$

so that $d_1 = -6$, $d_2 = -12$, *and* $d_3 = -48$ *(note the matrix is diagonal); therefore,* $(0, 0, 0)$ *is a saddle point. Now at* $(2, 0, 0)$, *the matrix is*

$$
\begin{pmatrix}
6 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 8
\end{pmatrix}
$$

so that $d_1 = 6$, $d_2 = 12$, *and* $d_3 = 96$; *therefore,* $(2, 0, 0)$ *is a minimum.*

7. (15 points) Find the maximum volume of a rectangular box contained in the ellipsoid $x^2 + 9y^2 + 4z^2 = 9$. [Hint: Explain why this box must have vertices lying on the ellipsoid and why one of these 'corners' must lie in the first octant. Call this point (x, y, z). Express the volume of the box in terms of this point and proceed.]

'Clearly', the box of maximum volume has all vertices lying on the ellipse (otherwise, it could be made larger) and has sides parallel to the coordinate axes. So one vertex is in the first octant, where $x > 0, y > 0, z > 0$. Let (x, y, z) be this vertex. The volume is given by $V(x, y, z) = (2x)(2y)(2z) = 8xyz$. The ellipsoid is given by $E(x, y, z) := x^2+9y^2+4z^2-9$.

$$\nabla V = \lambda \nabla E$$
$$\langle 8, yz, 8, xz, 8, xy\rangle = \lambda\langle 2x, 18y, 8z\rangle$$

Then we have a system of equations

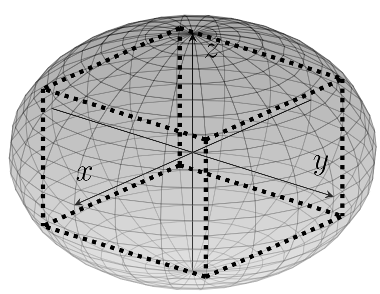
$$\begin{cases} 8yz = 2\lambda x \\ 8xz = 18\lambda y \\ 8xy = 8\lambda z \\ x^2 + 9y^2 + 4z^2 = 9 \end{cases}$$

Then we have $\lambda = \dfrac{8yz}{2x} = \dfrac{8xz}{18y} = \dfrac{8xy}{8z}$ *so that*

$$\lambda = \frac{4yz}{x} = \frac{4xz}{9y} = \frac{xy}{z}$$

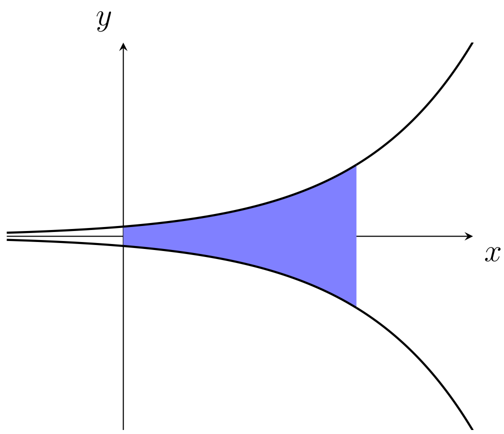
The second equality gives $36y^2z = 4x^2z$ so that $9y^2 = x^2$. The third equality gives $4xz^2 = 9xy^2$ so that $4z^2 = 9y^2$. Using $x^2 + 9y^2 + 4z^2 = 9$, we then find $9y^2 + 9y^2 + 9y^2 = 9$ so that $27y^2 = 9$ which means $y^2 = 1/3$. But then $x^2 = 9y^2 = 9(1/3) = 3$ and $4z^2 = 9y^2 = 9(1/3) = 3$, which means $z^2 = 3/4$. As $x, y, z > 0$, we have $x = \sqrt{3}$, $y = 1/\sqrt{3}$, and $z = \sqrt{3}/2$. But then the maximum volume is

$$V(\sqrt{3}, 1/\sqrt{3}, \sqrt{3}/2) = 8\sqrt{3} \cdot 1/\sqrt{3} \cdot \sqrt{3}/2 = 4\sqrt{3}$$



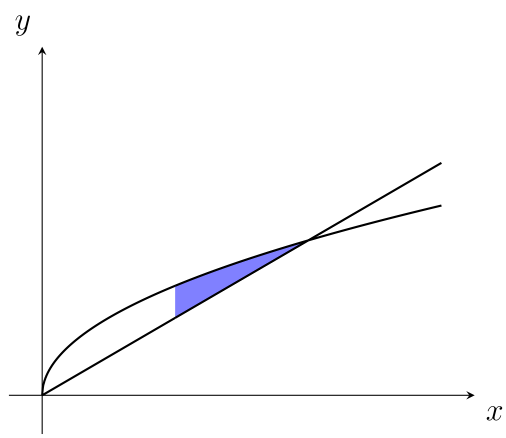
8. (12 points) Complete the following parts:

(a) Sketch the region of integration and evaluate $\int_0^2 \int_{-e^x}^{e^x} 3y^2\, dy\, dx$.



$$\begin{aligned}\int_0^2 \int_{-e^x}^{e^x} 3y^2\, dy\, dx &= \int_0^2 y^3 \Big|_{y=-e^x}^{y=e^x} dx \\ &= \int_0^2 (e^{3x} - (-e^{3x}))\, dx \\ &= 2\int_0^2 e^{3x}\, dx \\ &= 2 \cdot \frac{e^{3x}}{3}\Big|_{x=0}^{x=2} \\ &= \frac{2}{3}(e^6 - e^0) \\ &= \frac{2(e^6 - 1)}{3}\end{aligned}$$

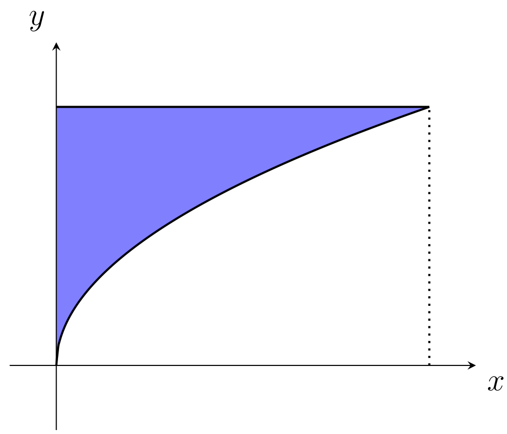
(b) Set up *but do not evaluate* the bounds for the integral $\iint_R xy\, dA$, where R is the region enclosed by $y = \frac{x}{2}$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.



$$\int_2^4 \int_{x/2}^{\sqrt{x}} xy\, dy\, dx$$

(c) Evaluate $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{10x}{1+y^5}\, dy\, dx$.

Note that $y=\sqrt{x} \Leftrightarrow y^2 = x$. *Then*



$$\begin{aligned}
\int_0^4 \int_{\sqrt{x}}^2 \frac{10x}{1+y^5}\, dy\, dx &= \int_0^2 \int_0^{y^2} \frac{10x}{1+y^5}\, dx\, dy \\
&= \int_0^2 \frac{1}{1+y^5} \int_0^{y^2} 10x\, dx\, dy \\
&= \int_0^2 \left.\frac{5x^2}{1+y^5}\right|_{x=0}^{x=y^2} dy \\
&= \int_0^2 \frac{5y^4}{1+y^5}\, dy \\
&= \ln|1+y^5| \Big|_{y=0}^{y=2} \\
&= \ln|1+2^5| - \ln|1+0| \\
&= \ln(33)
\end{aligned}$$