Math 397: Exam 3
Summer Session II – 2017
08/10/2017
145 Minutes

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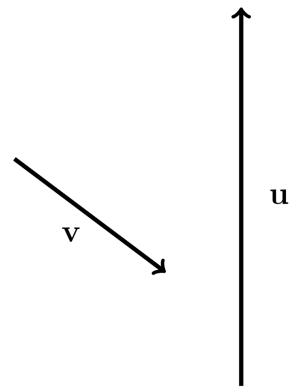
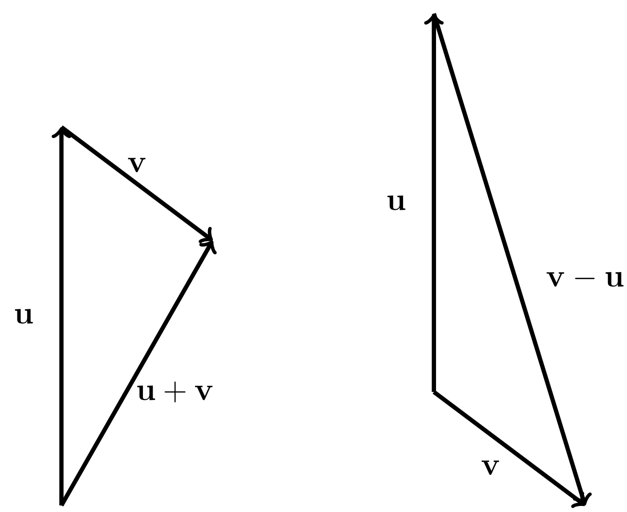
Write your name on the appropriate line on the exam cover sheet. This exam contains 22 pages (including this cover page) and 12 questions. Check that you have every page of the exam. ***You are to select any 10 of the first 12 problems, not including two bonus problems. If you attempt more than 10 of the first 12 problems, indicate clearly which 10 you wish to have graded.*** Answer these chosen questions in the spaces provided on the question sheets. Be sure to answer every part of each question and show all your work. If you run out of room for an answer, continue on the back of the page – being sure to indicate the problem number.

Question	Points	Score
1	0	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
8	0	
9	0	
10	0	
11	0	
12	0	
Total:	0	

1. (a) Given the vectors $\mathbf{u}$ and $\mathbf{v}$ below, sketch $\mathbf{u}+\mathbf{v}$ and $\mathbf{v}-\mathbf{u}$.





Let $\mathbf{a} = 2\,\hat{\mathbf{\imath}} - \hat{\mathbf{\jmath}} + 3\,\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{\imath}} - \hat{\mathbf{k}}$.

(b) Find $2\mathbf{b} - \mathbf{a}$.

$$2\mathbf{b} - \mathbf{a} = 2\langle 1, 0, -1\rangle - \langle 2, -1, 3\rangle = \langle 2, 0, -2\rangle - \langle 2, -1, 3\rangle = \langle 0, 1, -5\rangle$$

(c) Find a unit vector in the direction of $\mathbf{a}$.

$$\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\langle 2, -1, 3\rangle}{|\langle 2, -1, 3\rangle|} = \frac{\langle 2, -1, 3\rangle}{\sqrt{2^2 + (-1)^2 + 3^2}} = \frac{\langle 2, -1, 3\rangle}{\sqrt{14}} = \left\langle \frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$$

(d) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

We know that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$. *Now*

$$\begin{aligned} |\mathbf{a}| &= \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14} \\ |\mathbf{b}| &= \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{1 + 0 + 1} = \sqrt{2} \\ \mathbf{a} \cdot \mathbf{b} &= 2(1) + (-1)0 + 3(-1) = 2 + 0 - 3 = -1 \end{aligned}$$

But then this gives that

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{14}\sqrt{2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{28}}\right)$$

(e) Are $\mathbf{a}$ and $\mathbf{b}$ orthogonal? Justify your answer.

We know that $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal if and only if* $\mathbf{a} \cdot \mathbf{b} = 0$. *But from (d), we know that* $\mathbf{a} \cdot \mathbf{b} = -1 \neq 0$ *so that* $\mathbf{a}$ *and* $\mathbf{b}$ *are not orthogonal.*

2. Let $\mathbf{a} = 2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{k}}$.

(a) Find a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 3 \\ 1 & 0 & -1 \end{vmatrix} = \hat{\mathbf{i}}\,((-1)(-1) - 3(0)) - \hat{\mathbf{j}}\,(2(-1) - 3(1)) + \hat{\mathbf{k}}\,(2(0) - 1(-1)) \\ &= \hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} + \hat{\mathbf{k}} \\ &= \langle 1, 5, 1 \rangle\end{aligned}$$

(b) Find the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

$$|\mathbf{a} \times \mathbf{b}| = |\langle 1, 5, 1 \rangle| = \sqrt{1^2 + 5^2 + 1^2} = \sqrt{27} = 3\sqrt{3}$$

(c) Find the volume of the parallelepiped generated by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c} = \langle 1, -1, 4 \rangle$.

One can use the scalar triple product

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \\ 1 & -1 & 4 \end{vmatrix} = 0$$

or one could use the equivalent (and via (a) more convenient)

$$|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| = |\langle 1, -1, 4 \rangle \cdot \langle 1, 5, 1 \rangle| = |1(1) + (-1)5 + 4(1)| = |1 - 5 + 4| = 0$$

(d) Are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ coplanar?

From part (c), we know that the volume of the parallelepiped spanned by $\mathbf{a}$*,* $\mathbf{b}$*, and* $\mathbf{c}$ *is 0 which happens if and only if the parallelepiped is 'flat', i.e.* $\mathbf{a}$*,* $\mathbf{b}$*, and* $\mathbf{c}$ *are coplanar.*

3. Let $P(4,-6,4)$, $Q(7,-5,3)$, and $R(8,-3,2)$ be points in $\mathbb{R}^3$.

(a) Find the equation of the line through Q and R.

$$QR = \langle 8,-3,2\rangle - \langle 7,-5,3\rangle = \langle 1,2,-1\rangle$$

Then one possible solution to give the line is

$$\begin{aligned} l(t) &= (1,2,-1)t + (7,-5,3) \\ &= (t,2t,-t) + (7,-5,3) \\ &= (t+7, 2t-5, 3-t) \end{aligned}$$

Using the point R instead, one obtains another possibility, namely $l(t) = (t+8, 2t-3, 2-t)$.

(b) Find the equation of the plane perpendicular to the line from (a) and containing the point P.

Because the line from (a) is perpendicular to the plane, the direction vector for the line must be parallel to a normal vector for the plane. The plane contains the point $P(4,-6,4)$. Therefore, the equation of the plane is

$$\begin{aligned} \langle 1,2,-1\rangle \cdot \langle x-4, y-(-6), z-4\rangle &= 0 \\ 1(x-4) + 2(y+6) - (z-4) &= 0 \\ x - 4 + 2y + 12 - z + 4 &= 0 \\ x + 2y - z &= -12 \end{aligned}$$

(c) Find the distance from the point P to the line from (a).

The point Q is on the line from (a). Form the vector $PQ = \langle 7,-5,3\rangle - \langle 4,-6,4\rangle = \langle 3,1,-1\rangle$. Now we project onto the direction vector for the line

$$\text{proj}_{\langle 1,2,-1\rangle}\langle 3,1,-1\rangle = \frac{\langle 1,2,-1\rangle \cdot \langle 3,1,-1\rangle}{\langle 1,2,-1\rangle \cdot \langle 1,2,-1\rangle}\langle 1,2,-1\rangle = \frac{6}{6}\langle 1,2,-1\rangle = \langle 1,2,-1\rangle$$

But then the distance is

$$|\langle 3,1,-1\rangle - \langle 1,2,-1\rangle| = |\langle 2,-1,0\rangle| = \sqrt{2^2+(-1)^2+0^2} = \sqrt{5}$$

4. Let $f(x,y,z) = \dfrac{2x + ye^{3y} - z}{x}$.

(a) Find the equation of the tangent plane to $f(x,y,z)$ at the point $(1,0,-1)$.

$$\begin{aligned}\nabla f = \langle f_x,\, f_y,\, f_z \rangle &= \left\langle \frac{x(2) - (2x + ye^{3y} - z)}{x^2},\ \frac{1}{x}(e^{3y} + 3ye^{3y}),\ \frac{-1}{x} \right\rangle \\ &= \left\langle \frac{z - ye^{3y}}{x^2},\ \frac{e^{3y}(1+3y)}{x},\ \frac{-1}{x} \right\rangle\end{aligned}$$

$$\nabla f(1,0,-1) = \left\langle \frac{-1 - 0e^0}{1^2},\ \frac{e^0(1+3(0))}{1},\ \frac{-1}{1} \right\rangle = \langle -1, 1, -1 \rangle$$

Then the equation of the plane is

$$\begin{aligned}\langle -1, 1, -1\rangle \cdot \langle x - 1, y - 0, z - (-1)\rangle &= 0 \\ -1(x-1) + 1(y-0) - 1(z+1) &= 0 \\ -x + 1 + y - z - 1 &= 0 \\ y - x - z &= 0\end{aligned}$$

(b) Find f_{xy}.

$$f_{xy} = f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial y}\left(\frac{e^{3y}(1+3y)}{x}\right) = \frac{-e^{3y}(1+3y)}{x^2}$$

(c) Find $\dfrac{\partial^2 f}{\partial y \partial z}$.

$$\frac{\partial^2}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial y}\left(\frac{-1}{x}\right) = 0$$

5. Let $f(x, y, z) = e^{x^2+y^2+z^2}$.

(a) Find the rate of change of $f(x, y, z)$ in the direction of $\mathbf{u} = -\hat{\imath} + \hat{\jmath} - \hat{\mathbf{k}}$ at the point $(0, 2, -1)$.

$$\nabla f = \left\langle 2xe^{x^2+y^2+z^2}, 2ye^{x^2+y^2+z^2}, 2ze^{x^2+y^2+z^2} \right\rangle$$

$$\nabla f(0, 2, -1) = \left\langle 2(0)e^{0^2+2^2+(-1)^2}, 2(2)e^{0^2+2^2+(-1)^2}, 2(-1)e^{0^2+2^2+(-1)^2} \right\rangle = \langle 0, 4e^5, -2e^5 \rangle$$

$$\nabla f(0, 2, -1) \cdot \frac{\mathbf{u}}{|\mathbf{u}|} = \langle 0, 4e^5, -2e^5 \rangle \cdot \frac{\langle -1, 1, -1 \rangle}{\sqrt{(-1)^2 + 1^2 + (-1)^2}} = \frac{6e^5}{\sqrt{3}} = 2\sqrt{3}e^5$$

(b) What is the direction of maximum increase for the function $f(x, y, z)$ at the point $(0, 2, -1)$? What is the direction of maximum decrease for the function $f(x, y, z)$ at the point $(0, 2, -1)$?

The direction of maximum increase is $\langle 0, 4e^5, -2e^5 \rangle$ while the rate of maximum decrease is $-\langle 0, 4e^5, -2e^5 \rangle = \langle 0, -4e^5, 2e^5 \rangle$. Note that since $\langle 0, 2, -1 \rangle$ and $\langle 0, -2, 1 \rangle$ point in the same direction as the given vectors, respectively, these can equally be given as the direction of maximum increase/decrease, respectively.

(c) Find the rate of change for $f(x, y, z)$ at the directions you found in (b).

The rate of maximum increase is

$$|\langle 0, 4e^5, -2e^5 \rangle| = \sqrt{0^2 + (4e^5)^2 + (-2e^5)^2} = \sqrt{20e^{10}} = 2e^5\sqrt{5}$$

while the rate of maximum decrease is

$$-|\langle 0, 4e^5, -2e^5 \rangle| = -\sqrt{0^2 + (4e^5)^2 + (-2e^5)^2} = -\sqrt{20e^{10}} = -2e^5\sqrt{5}$$

6. Choose either (a) or (b) to complete. *You* ***do not*** *need to do both. Choose only of of the two and complete it.*

   (a) Find and classify the critical points for the function $f(x, y, z) = e^z(z^2 - y^2 - 2x^2)$.

   (b) Find the maximum and minimum values of $F(x, y, z) = 2x^3 + y^3 + 2z^{3/2}$ if x, y, and z satisfy $x^4 + y^4 + z^2 = 33$ and $xyz \neq 0$. [Note: $xyz \neq 0$ simply says that none of x, y, or z are zero.]

(a) We have

$$\begin{cases} f_x = -4xe^z \\ f_y = -2ye^z \\ f_z = e^z(z^2 - z^2 - x^2) + 2ze^z = e^z(z^2 + 2z - 2x^2 - y^2) \end{cases}$$

Setting $f_x = 0$ gives $-4xe^z = 0$ which implies $e^z = 0$, impossible, or $-4x = 0$ so that $x = 0$. Setting $f_y = 0$ yields $-2ye^z = 0$ so that $e^z = 0$, impossible, or $-2y = 0$ so that $y = 0$. Finally, setting $f_z = 0$ implies that (using the fact that $x = y = 0$) $e^z(z^2 + 2z - 2x^2 - y^2) = e^z(z^2 + 2z) = e^z z(z + 2) = 0$, which implies $e^z = 0$, impossible, $z = 0$, or $z + 2 = 0$ so that $z = -2$. Therefore, we have two critical points: $(0, 0, 0)$ and $(0, 0, -2)$. The Hessian is

$$\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} -4e^z & 0 & -4xe^z \\ 0 & -2e^z & -2ye^z \\ -4xe^z & -2ye^z & e^z(2 - 2x^2 - y^2 + 4z + z^2) \end{pmatrix}$$

At the point $(0, 0, 0)$, the Hessian is

$$\begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Then $d_1 = -4 < 0$, $d_2 = (-4)(-2) = 8 > 0$, and $d_3 = (-4)(-2)(2) = 16 > 0$ so that $(0, 0, 0)$ is a saddle. At the point $(0, 0, -2)$, the Hessian is

$$\begin{pmatrix} -4e^2 & 0 & 0 \\ 0 & -2e^2 & 0 \\ 0 & 0 & -\dfrac{2}{e^2} \end{pmatrix}$$

Then $d_1 = -4e^2 < 0$, $d_2 = (-4e^2)(-2e^2) = 8e^4 > 0$, and $d_3 = (-4e^2)(-2e^2)(-2e^{-2}) = -16e^2 < 0$ so that $(0, 0, -2)$ is a maximum.

(b) *Let* $G(x,y,z) = x^4 + y^4 + z^2 - 33$*. We have*

$$\nabla F = \langle 6x^2, 3y^2, 3z^{1/2} \rangle$$
$$\nabla G = \langle 4x^3, 4y^3, 2z \rangle$$

Lagrange's Theorem tells us there is a $\lambda \neq 0$ *such that* $\nabla F = \lambda \nabla G$*. This gives the system of equations*

$$\begin{cases} 6x^2 = 4\lambda x^3 \\ 3y^2 = 4\lambda y^3 \\ 3z^{1/2} = 2\lambda z \end{cases}$$

Now $x, y, z \neq 0$ *so that this is equivalent to*

$$\begin{cases} 3 = 2\lambda x \\ 3 = 4\lambda y \\ 3 = 2\lambda z^{1/2} \end{cases}$$

But then we have

$$\lambda = \frac{3}{2x} = \frac{3}{4y} = \frac{3}{2z^{1/2}}$$

The second and third equalities give $\dfrac{3}{2x} = \dfrac{3}{4y}$ *so that* $x = 2y$*. The third and fourth equalities give* $\dfrac{3}{4y} = \dfrac{3}{2z^{1/2}}$ *so that* $z^{1/2} = 2y = x$ *which also implies* $z = 4y^2$*. But then*

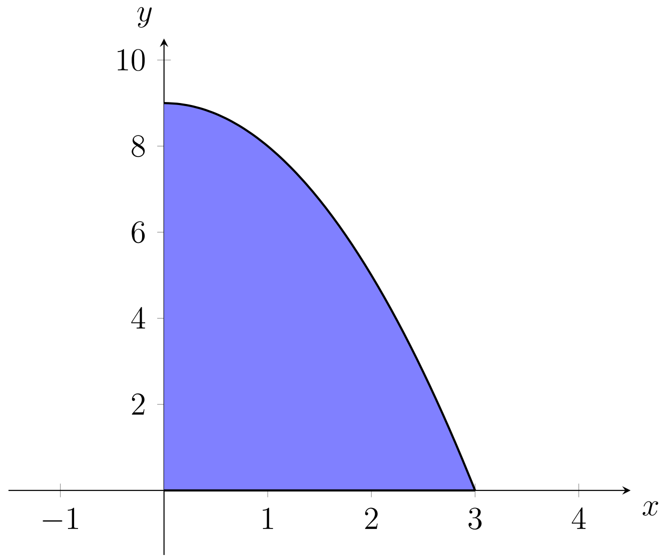
$$33 = x^4 + y^4 + z^2 = (2y)^4 + y^4 + (4y^2)^2 = 16y^4 + y^2 + 16y^4 = 33y^4$$

So that $y^4 = 1$ *and therefore* $y = \pm 1$*. If* $y = 1$ *then* $x = 2$ *and* $z = 4$*, giving the point* $(2, 1, 4)$*. If* $y = -1$*, we have* $x = -2$ *and* $z = 4$ *giving the point* $(-2, -1, 4)$*. Then as* $F(2, 1, 4) = 33$ *and* $F(-2, -1, 4) = -1$*,* $F(2, 1, 4) = 33$ *is the maximum value and* $F(-2, -1, 4) = -1$ *is the minimum value for* $F(x, y, z)$ *given the constraints* $G(x, y, z) = 0$ *and* $xyz \neq 0$*.*

7. Complete the following parts:

(a) Compute the following integral:

$$\int_0^3 \int_0^{9-x^2} \frac{xe^{3y}}{9-y}\, dy\, dx$$



The region, shown above, has boundary curves given by $x = 0$, $y = 0$, and $y = 9 - x^2$. Note that if $y = 9 - x^2$ then $x = \pm\sqrt{9-y}$ but as $x \geq 0$ in our region, we have $x = \sqrt{9-y}$. Then we have

$$\begin{aligned}
\int_0^3 \int_0^{9-y^2} \frac{xe^{3y}}{9-y}\, dy\, dx &= \int_0^9 \int_0^{\sqrt{9-y}} \frac{xe^{3y}}{9-y}\, dx\, dy \\
&= \int_0^9 \left. \frac{x^2 e^{3y}}{2(9-y)} \right|_{x=0}^{x=\sqrt{9-y}} dy \\
&= \frac{1}{2} \int_0^9 e^{3y}\, dy \\
&= \frac{1}{2} \cdot \left. \frac{e^{3y}}{3} \right|_{y=0}^{y=9} \\
&= \frac{1}{6} \cdot \left. e^{3y} \right|_{y=0}^{y=9} \\
&= \frac{e^{27} - 1}{6}
\end{aligned}$$

(b) Set up completely as possible *but do not integrate* any integral which would compute the volume of the region enclosed by $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.

Note that these surfaces are both paraboloids. They intersect at $x^2 + z^2 = 8 - x^2 - z^2$ *which implies* $2x^2 + 2z^2 = 8$ *so that* $x^2 + z^2 = 4$*, which is a circle in the* xz*–plane of radius 2.*

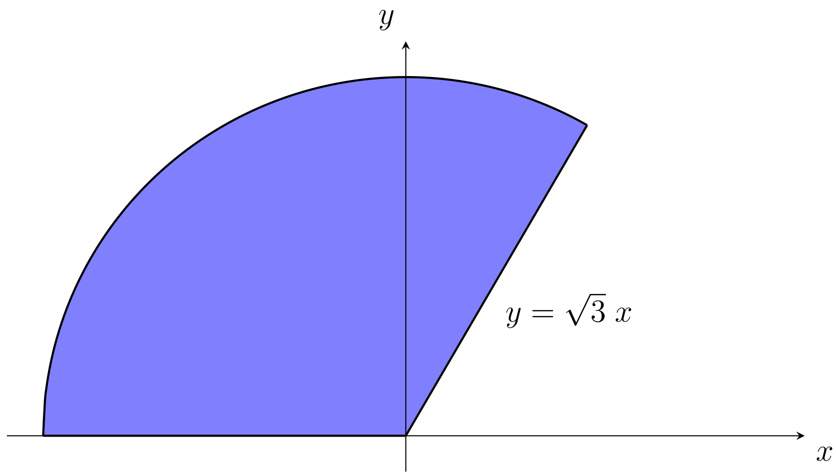
$$V = \iiint_R dV = \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{x^2+z^2}^{8-x^2-z^2} dy\, dx\, dz$$

8. Complete the following parts:

   (a) Change the integral given below into an integral in polar coordinates. Be sure to set up the integral completely as possible *but do not evaluate the integral*.

$$\iint_R \cos(x^2 + y^2)\, dA$$

The region R is the shaded region portion of the unit circle in the figure below:



Note that since we have a unit circle, $0 \leq r \leq 1$. *Clearly, the terminal* θ *is* π. *We only need the initial angle,* θ_0. *However, we know that a point on the circle is* $(r\cos\theta, r\sin\theta)$, *which for this circle is* $(\cos\theta, \sin\theta)$. *Then we know at the farthest most right point is the point* $(x, y) = (x, \sqrt{3}x) = (\cos\theta_0, \sin\theta_0)$. *But* $x^2 + y^2 = 1$ *so that* $x^2 + 3x^2 = 1$ *so that* $x = 1/2$. *But* $1/2 = x = \cos\theta_0$ *so that* $\theta_0 = \frac{\pi}{6}$. *Alternatively, observe the initial angle is given by* $\theta_0 = \arctan(y/x) = \arctan(\sqrt{3}x/x) = \arctan(\sqrt{3}) = \frac{\pi}{6}$. *Then we have*

$$\int_{\pi/6}^{\pi} \int_0^1 r\cos r^2\, dr\, d\theta$$

(b) Change the integral given below into an integral in cylindrical coordinates. Be sure to set up the integral completely as possible *but do not evaluate the integral*.

$$\iiint_R \left(2+\sqrt{x^2+y^2}\right)\,dV$$

The region R is given by $R = \left\{(x,y,z) \colon \sqrt{x^2+y^2} \leq \frac{z}{2} \leq 3\right\}$.

Observe $\sqrt{x^2+y^2} \leq \dfrac{z}{2} \leq 3$ *implies that* $2\sqrt{x^2+y^2} \leq z \leq 3$. *Then the 'top' of the region is* $z=3$ *and the 'side' of the region is* $z^2 = 4(x^2+y^2)$. *But then* $9 = 4(x^2+y^2)$ *so that* $x^2+y^2 = \frac{9}{4}$. *But then the integral is*

$$\int_0^{2\pi}\int_0^{3/2}\int_{2\sqrt{r^2}}^{3} \left(2+\sqrt{r^2}\right) r\;dz\;dr\;d\theta = \int_0^{2\pi}\int_0^{3/2}\int_{2r}^{3} (2r+r^2)\;dz\;dr\;d\theta$$

(c) Change the integral given below into an integral in spherical coordinates. Be sure to set up the integral completely as possible *but do not evaluate the integral*.

$$\iiint_R \frac{y}{\sqrt{x^2+y^2+z^2}}\;dV$$

The region R is the collection of points between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$, where $y>0$ and $z>0$.

$$\int_0^{\pi/2}\int_0^{\pi}\int_1^2 \frac{\rho\sin\phi\sin\theta}{\sqrt{\rho^2}} \cdot \rho^2\sin\phi\;d\rho\;d\theta\;d\phi = \int_0^{\pi/2}\int_0^{\pi}\int_1^2 \rho^2\sin^2\phi\sin\theta\;d\rho\;d\theta\;d\phi$$

9. Let C be the directed line segment from the point $(-2, 1)$ to the point $(1, 3)$.

(a) Compute $\displaystyle\int_C (x + 2y)\ dx$.

We have $r(t) = (-2, 1)(1 - t) + (1, 3)t = (3t - 2, 2t + 1)$ *for* $0 \leq t \leq 1$ *so that* $x = 3t - 2$, $dx = 3\,dt$, *and* $y = 2t + 1$. *Then* $x + 2y = (3t - 2) + 2(2t + 1) = 7t$. *Therefore,*

$$\int_C (x + 2y)\ dx = \int_0^1 7t \cdot 3\ dt = 21 \int_0^1 t\ dt = \frac{21}{2}$$

(b) Compute $\displaystyle\int_C (x + 2y)\ dy$.

We have $r(t) = (-2, 1)(1 - t) + (1, 3)t = (3t - 2, 2t + 1)$ *for* $0 \leq t \leq 1$ *so that* $x = 3t - 2$, $y = 2t + 1$, *and* $dy = 2\,dt$. *Then* $x + 2y = (3t - 2) + 2(2t + 1) = 7t$. *Therefore,*

$$\int_C (x + 2y)\ dy = \int_0^1 7t \cdot 2\ dt = 14 \int_0^1 t\ dt = 7$$

(c) Compute $\displaystyle\int_C (x + 2y)\ ds$

We have $r(t) = (-2, 1)(1 - t) + (1, 3)t = (3t - 2, 2t + 1)$ *for* $0 \leq t \leq 1$ *so that* $x = 3t - 2$, $dx = 3\,dt$, $y = 2t + 1$, *and* $dy = 2\,dt$ *so that* $ds = \sqrt{3^2 + 2^2} = \sqrt{13}$. *Then* $x + 2y = (3t - 2) + 2(2t + 1) = 7t$. *Therefore,*

$$\int_C (x + 2y)\ ds = \int_0^1 7t \cdot \sqrt{13}\ dt = 7\sqrt{13} \int_0^1 t\ dt = \frac{7\sqrt{13}}{2}$$

10. Complete the following parts:

(a) Compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and $\mathbf{r}(t) = (t, 3t^2, 2t^3)$ for $0 \le t \le 1$.

We have $\mathbf{F}(\mathbf{r}(t)) = \langle t, 3t^2, 2t^3\rangle$ *and* $\mathbf{r}'(t) = \langle 1, 6t, 6t^2\rangle$. *Therefore,*

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle t, 3t^2, 2t^3\rangle \cdot \langle 1, 6t, 6t^2\rangle \; dt \\ &= \int_0^1 t + 18t^3 + 12t^5 \; dt \\ &= \left(\frac{t^2}{2} + \frac{18t^4}{4} + \frac{12t^6}{6}\right)\Bigg|_{t=0}^{t=1} \\ &= \frac{1}{2} + \frac{18}{4} + \frac{12}{6} \\ &= \frac{1}{2} + \frac{9}{2} + 2 \\ &= 7\end{aligned}$$

(b) Compute

$$\oint_C -y \; dx + x \; dy$$

where C is the circle of radius 3 centered at the origin, oriented counterclockwise.

The circle can be parametrized by $\mathbf{r}(t) = (3\cos t, 3\sin t)$ *for* $0 \le t \le 2\pi$. *Therefore,* $x = 3\cos t$, $dx = -3\sin t\, dt$, $y = 3\sin t$, *and* $dy = 3\cos t\, dt$. *Then we have*

$$\begin{aligned}\oint_C -y \; dx + x \; dy &= \int_0^{2\pi} -3\sin t \cdot -3\sin t \; dt + 3\cos t \cdot 3\cos t \; dt \\ &= \int_0^{2\pi} 9\sin^2 t + 9\cos^2 t \; dt \\ &= 9\int_0^{2\pi} \; dt \\ &= 18\pi\end{aligned}$$

11. Let $\mathbf{F} = (2xy+1)\,\hat{\mathbf{i}} + (x^2-1)\,\hat{\mathbf{j}}$.

(a) Compute div $\mathbf{F}$.

$$\textit{div } \mathbf{F} = \frac{\partial}{\partial x}(2xy+1) + \frac{\partial}{\partial y}(x^2-1) = 2y + 0 = 2y$$

(b) Use the curl to show that $\mathbf{F}$ is conservative.

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+1 & x^2-1 & 0 \end{vmatrix} \\
&= \hat{\mathbf{i}}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x^2-1)\right) - \hat{\mathbf{j}}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2xy+1)\right) \\
&\qquad + \hat{\mathbf{k}}\left(\frac{\partial}{\partial x}(x^2-1) - \frac{\partial}{\partial y}(2xy+1)\right) \\
&= \hat{\mathbf{i}}\,(0-0) - \hat{\mathbf{j}}\,(0-0) + \hat{\mathbf{k}}\,(2x-2x) \\
&= \langle 0,0,0\rangle \\
&= \mathbf{0}
\end{aligned}$$

Because $\mathbf{F}$ is clearly C^2 on $\mathbb{R}^2$, which is simply connected, '$\nabla \times \mathbf{F} = \mathbf{0}$' implies that $\mathbf{F}$ is conservative.

(c) Find a potential function for $\mathbf{F}$.

If $\mathbf{F} = \nabla f$, then $\mathbf{F} = \langle f_x,\ f_y \rangle$ so that $f_x = 2xy + 1$ and $f_y = x^2 - 1$. But then

$$f(x,y) = \int f_x \; dx = \int (2xy + 1) \; dx = x^2 y + x + C(y)$$

where $C(y)$ is a function of y alone. But then we must have

$$x^2 - 1 = f_y = \frac{\partial}{\partial y}(x^2 y + x + C(y)) = x^2 + C'(y)$$

which implies that $C'(y) = -1$ so that $C(y) = -y + C$, where C is some constant of integration. Therefore, $f(x,y) = x^2 y + x - y + C$ is a potential function for $\mathbf{F}$.

(d) Compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C : [0,1] \to \mathbb{R}^2$ is the path given by

$$\mathbf{r}(t) = \left(e^{t^2 - t} + \sin\left(\pi \cos\left(\frac{\pi t}{2}\right)\right) + 2t,\ \frac{1}{t^2 + 2t - 4} - \sin(\pi t) + \frac{1-t}{4} \right)$$

Because we know that $\mathbf{F}$ is conservative, we know that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0))$$

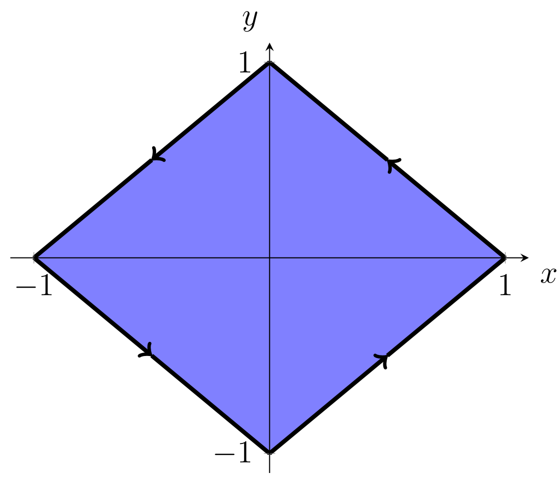
Now $\mathbf{r}(1) = (3, -1)$ and $\mathbf{r}(0) = (1, 0)$ so that

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= f(3,-1) - f(1,0) = (3^2(-1) + 3 - (-1) + C) - (1^2(0) + 1 - 0 + C) \\
&= -9 + 3 + 1 + C - 0 - 1 + 0 - C \\
&= -6
\end{aligned}$$

12. Compute

$$\oint_C (xy^2 + 2y^3 + y)\, dx + (x^2y + 6xy^2 + 10x)\, dy$$

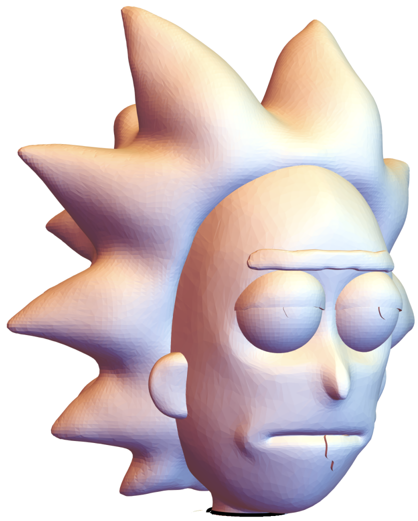
where C is the boundary of the square, oriented counterclockwise, shown below

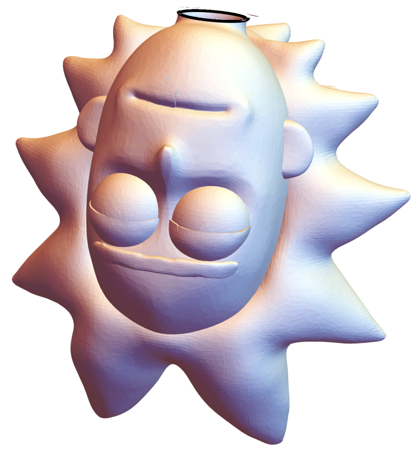


Let R denote the square indicated by the figure. The boundary of this square, ∂R, is a simple closed curve that is piecewise smooth so that Green's Theorem applies. Therefore,

$$\begin{aligned}
&\oint_C (xy^2 + 2y^3 + y)\, dx + (x^2y + 6xy^2 + 10x)\, dy = \\
&\iint_R \frac{\partial}{\partial x}(x^2y + 6xy^2 + 10x) - \frac{\partial}{\partial y}(xy^2 + 2y^3 + y)\, dA = \\
&\iint_R (2xy + 6y^2 + 10) - (2xy + 6y^2 + 1)\, dA = \\
&9 \iint_R \, dA = 9 \cdot \textit{Area}(R) = 9 \cdot \left(\sqrt{1^2 + 1^2}\right)^2 = 18
\end{aligned}$$

Bonus 1: Suppose S is the surface consisting of 214,252 polygons that are smoothly connected to form the visage of the scientist Rick Sanchez—also known as Pickle Rick, where the equation of the base of the neck (highlighted on the surface in dark black) is the circle $4x^2 + 4y^2 = 9$ in the plane $z = 47$ (see the figures below).[1]





If $\mathbf{F} = (x - 2y)\,\hat{\mathbf{\imath}} + (2x - y)\,\hat{\mathbf{\jmath}} + (xz + yz - x\sin(yz))\,\hat{\mathbf{k}}$, calculate

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

Be sure to justify all steps in your calculations and any theorems used thoroughly.

Observe that S and $\mathbf{F}$ satisfy the conditions for Stokes' Theorem. Then by the theorem, we have

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{S}$$

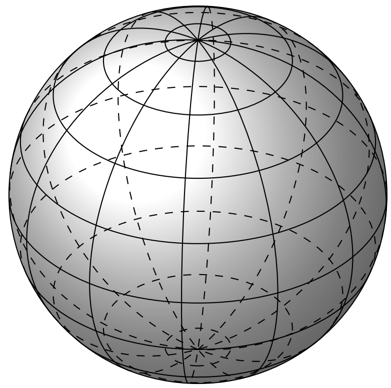
But then by Stokes' Theorem, the given surface integral must be equal to $\oint_{\partial S'} \mathbf{F} \cdot d\mathbf{S}'$ for any *surface S', which satisfies the conditions for Stokes' Theorem, such that $\partial S = \partial S'$. Observe that the disk $D = \{(x, y, z)\colon 4x^2 + 4y^2 = 9, z = 47\}$ satisfies this condition (its boundary is clearly the given circle at $z = 47$). We calculate the integral for this surface instead, i.e. $S' = D$. Now parametrize the boundary of this disk by $\mathbf{r}(t) = \left(\frac{3}{2}\cos t, \frac{3}{2}\sin t, 47\right)$ for $0 \leq t \leq 2\pi$. Then $\mathbf{r}'(t) = \left\langle -\frac{3}{2}\sin t, \frac{3}{2}\cos t, 0\right\rangle$. Write $\mathbf{F}$ as*

[1] ChaosCoreTech. 2017, July. Rick Sanchez [Rick and Morty]. https://pinshape.com/items/33235-3d-printed-rick-sanchez-rick-and-morty.
Rick and Morty. Warner Bros. Television. Warner Bros. Television Distribution. July 2017. Television.

$\mathbf{F} = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$. *Then we have*

$$\begin{aligned}
\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}' \\
&= \oint_{\partial S'} \mathbf{F} \cdot d\mathbf{S}' \\
&= \int_0^{2\pi} \langle M(\mathbf{r}(t)), N(\mathbf{r}(t)), P(\mathbf{r}(t))\rangle \cdot \left\langle -\frac{3}{2}\sin t, \frac{3}{2}\cos t, 0 \right\rangle \, dt \\
&= \int_0^{2\pi} \left(\frac{3}{2}\cos t - 2 \cdot \frac{3}{2}\sin t\right)\left(-\frac{3}{2}\sin t\right) + \\
&\qquad\qquad \left(2 \cdot \frac{3}{2}\cos t - \frac{3}{2}\sin t\right)\left(\frac{3}{2}\cos t\right) + 0 \, dt \\
&= \frac{9}{2}\int_0^{2\pi} \cos^2 t + \sin^2 t - 9\sin t\cos t \, dt \\
&= \frac{9}{2}\int_0^{2\pi} dt + \frac{9}{2}\int_0^{2\pi} \frac{\sin 2t}{2} \, dt \\
&= \frac{9}{2}\cdot 2\pi + 0 \\
&= 9\pi
\end{aligned}$$

Bonus 2: Verify the Divergence Theorem for the vector field $\mathbf{F} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ over the sphere S of radius R centered at the origin, i.e. $x^2 + y^2 + z^2 = R^2$.



Because the solid sphere, which we shall denote as R, is a bounded solid region in $\mathbb{R}^3$ whose boundary is smooth, closed, and orientable and $\mathbf{F}$ is C^∞ on $\mathbb{R}^3$, the Divergence Theorem/Gauss' Theorem applies so that

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_R \nabla \cdot \mathbf{F}\; dV$$

To verify, we compute both integrals. We begin with the left integral. Parametrize the sphere by

$$\mathbf{X}(s,t) = (R\cos s \sin t, R\sin s \sin t, R\cos t)$$

for $0 \leq s \leq 2\pi$ and $0 \leq t \leq \pi$. Then we have

$$\begin{aligned} \mathbf{T}_s &= \langle -R\sin s \sin t, R\cos s \sin t, 0\rangle \\ \mathbf{T}_t &= \langle R\cos s \cos t, R\sin s \sin t, -R\cos t\rangle \end{aligned}$$

Then $\mathbf{N}(s,t) = \mathbf{T}_s \times \mathbf{T}_t = -R^2 \sin t \langle \cos s \sin t, \sin s \sin t, \cos t\rangle$. But this vector points inwards so we must negate the product. Then

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^\pi \int_0^{2\pi} \langle R\cos s \sin t, R\sin s \sin t, R\cos t\rangle \\ &\qquad\qquad \cdot \left(R^2 \sin t \langle \cos s \sin t, \sin s \sin t, \cos t\rangle\right)\; ds\; dt \\ &= R^3 \int_0^\pi \int_0^{2\pi} \sin t(\cos^2 s \sin^2 t + \sin^2 s \sin^2 t + \cos^2 t)\; ds\; dt \\ &= R^3 \int_0^\pi \int_0^{2\pi} \sin t\; ds\; dt \\ &= 2\pi R^3 \int_0^\pi \sin t\; dt \\ &= 4\pi R^3 \end{aligned}$$

Note that one could also use

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_R \mathbf{F}(\mathbf{X}(s,t))\cdot \mathbf{N}(s,t)\; ds\; dt\\ &= \iint_R \mathbf{F}(\mathbf{X}(s,t))\cdot (\|N(s,t)\|\, \mathbf{n}(s,t))\; ds\; dt\\ &= \iint_S (\mathbf{F}\cdot\mathbf{n})\; dS\end{aligned}$$

where $\mathbf{n}$ *is the unit outward normal. Then use the fact that the sphere has unit outward normal* $\mathbf{n} = \dfrac{\langle x,y,z\rangle}{R}$ *so that on* S *we have* $\mathbf{F}\cdot\mathbf{n} = \dfrac{x^2+y^2+z^2}{R} = \dfrac{R^2}{R} = R$*. We only need integrate over the upper half sphere, say* H*, due to symmetry. But then we have, using* $dS = R^2\sin\phi\; d\theta\; d\phi$*,*

$$\begin{aligned}\oiint_S (\mathbf{F}\cdot\mathbf{n})\; dS &= 2R\iint_H \; dS\\ &= 2R\int_0^{2\pi}\int_0^{\pi/2} R^2\sin\phi\; d\phi\; d\theta\\ &= 2R^3\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2}\sin\phi\; dr\right)\\ &= 2R^3\cdot 2\pi\cdot 1\\ &= 4\pi R^3\end{aligned}$$

Alternatively, observe $\iint_S (\mathbf{F}\cdot\mathbf{n})\; dS = R\iint_S \; dS = R\cdot$ *surface area* S*. But the surface area of a sphere is* $4\pi R^2$ *so that we have* $4\pi R^3$ *as expected. Now we compute the right side. Observe*

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

But then

$$\begin{aligned}\iiint_R \nabla\cdot\mathbf{F}\; dV &= \iiint_R 3\; dV\\ &= 3\iiint_V \; dV\\ &= 3\int_0^{2\pi}\int_0^{\pi}\int_0^R \rho^2\sin\phi\; d\rho\; d\phi\; d\theta\\ &= 3\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\; d\phi\right)\left(\int_0^R \rho^2\; d\rho\right)\\ &= 3\cdot 2\pi\cdot 2\cdot\frac{R^3}{3}\\ &= 4\pi R^3\end{aligned}$$

Alternatively, recognize $3\iiint_V \; dV = 3$ *Volume(sphere)* $= 3\cdot\dfrac{4}{3}\pi R^3 = 4\pi R^3$.