Multivariable Derivatives

In Calculus, our notion of change is most often the derivative. Unlike in your first Calculus course, there are several notions of change in Multivariable Calculus: derivatives, partial derivatives, and directional derivatives. In this project, we will examine the complexity of the relationship between these different notions of change.

1 Partially Differentiable but not Differentiable

Define the function

$$f(x,y) := \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

(a) Use software of your choice to plot $f(x,y)$ 'near' the origin.

(b) Find f_x and f_y.

(c) Using the definition of the partial derivative, find $f_x(0,0)$ and $f_y(0,0)$.

(d) Show that $f(x,y)$ is not differentiable at the origin. [Hint: Show the function is not continuous at the origin.]

(e) To see how the example was created, convert $f(x,y)$ to polar coordinates. Call this new function $g(r,\theta)$. Use $g(r,\theta)$ to explain why $f(x,y)$ is not continuous at $(0,0)$.

2 Continuous but not Differentiable

To see that a function can be continuous, have all the partial derivatives exist, and the function is still not differentiable, consider the function $h(x,y) = (xy)^{1/3}$.

(a) Find h_x and h_y.

(b) Find at which points h_x and h_y are zero and at which points they are not defined. Describe these regions geometrically.

(c) Show that $f_x(0,0)$ and $f_y(0,0)$ both exist.

(d) Show that f is not differentiable at $(0,0)$. [Hint: Consider the directional derivative along $y = x$.]

3 Directional Derivatives but not Differentiable

We can go even further. There are functions for which the directional derivative at $(0,0)$ exists in all directions but the function is not differentiable (or even continuous) there. Consider the function

$$f(x,y) = \begin{cases} \dfrac{yx^2}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

(a) Show that $f(x, y)$ is not continuous at $(0, 0)$.

(b) Use the definition of a directional derivative to show that the directional derivative exists for any unit vector $\mathbf{u} = \langle a, b \rangle$.

(c) Show that $D_u f(0, 0) \neq \nabla f(0, 0) \cdot \mathbf{u}$.

Evaluation

Complete the following survey by rating each problem. Each area will be rated on a scale of 1 to 5. For interest, 1 is “mind-numbing" while a 5 is “mind-blowing". For difficulty, 1 is “trivial/routine" while 5 is “brutal." For learning, 1 means “nothing new" while 5 means "profound awakening". Then you to estimate the amount of time you spent on each problem (in minutes).

	Interest	Difficulty	Learning	Time Spent
Partially Differentiable but not Differentiable				
Continuous but not Differentiable				
Directional Derivatives but not Differentiable				

Indicate whether you believe this project was helpful in mastering the course material and/or if it was helpful in developing a deeper understanding of the subject. Also, indicate whether you think this project should be given to future Calculus III students.

	Yes	No
Helpful for the Course		
Helpful in Learning the Subject		
Assign Again		

Finally, you may write any comments, thoughts, or suggestions in the space below.