Integrals and Summations

1 The Basel Problem

The sum of this series has a rich history. It was first posed by Pietro Mengoli in 1644. However, the problem went unsolved for nearly a century. It was popularized in 1689 by Jakob Bernoulli. The problem subsequently became known as the Basel Problem (where was Jakob Bernoulli was then living). None of the Bernoulli brothers were able to solve the problem. Leibniz also failed to solve the problem. It was first solved by Leonhard Euler in 1734. [One should note that the method here is not due to Euler.] Solving this problem brought the then 28 year old Euler—destined to become regarded as the greatest mathematician to ever live – great fame. In fact during his lifetime, Euler gave at least 4 different proofs of this fact. A survey of mathematicians declared this to be the fifth most beautiful equation in all of mathematics.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

We will evaluate the sum by evaluating the following integral two different ways: $I := \int_0^1 \int_0^1 \frac{dx\,dy}{1-xy}$

(a) Find the Taylor Series for $\frac{1}{1-x}$.

(b) Use the previous part to find the sum $\sum_{n=0}^{\infty} (xy)^n = 1 + xy + x^2y^2 + x^3y^3 + \cdots$.

(c) Use the previous part to show that I is the same as

$$\sum_{n=0}^{\infty} \left(\int_0^1 x^n \, dx \right) \left(\int_0^1 y^n \, dy \right)$$

(d) Evaluate the integral and index shift to show that I is $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$, where ζ is the Riemann–Zeta function.

(e) Consider the substitution $u = \frac{x+y}{2}$ and $v = \frac{y-x}{2}$. Sketch the original region of integration for I and the region after the given substitution. Is there any symmetry in the new region?

(f) Use the substitution from the previous part, along with the symmetry of the region, to show that

$$I = 4 \int_0^{1/2} \left(\int_0^u \frac{dv}{1-u^2+v^2} \right) du + 4 \int_{1/2}^1 \left(\int_0^{1-u} \frac{dv}{1-u^2+v^2} \right) du$$

(g) Evaluate the integral $\int \frac{dx}{a^2+x^2}$.

(h) Use the previous to parts to show that

$$I = 4\int_0^{1/2} \frac{1}{\sqrt{1-u^2}} \arctan\left(\frac{u}{\sqrt{1-u^2}}\right)\, du + 4\int_{1/2}^{1} \frac{1}{\sqrt{1-u^2}} \arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right)\, du$$

(i) Define $g(u) := \arctan\left(\frac{u}{\sqrt{1-u^2}}\right)$ and $h(u) := \arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right)$. Find $g'(u)$ and $h'(u)$.

(j) In terms of $f(a)$ and $f(b)$, $\int_a^b f(x)f'(x)\, dx$.

(k) Use the previous two parts to evaluate the integrals in (h) is $2g(1/2)^2 + 4h(1/2)^2 - 2g(0)^2 - 4h(1)^2$.

(l) Show that the sum is $\frac{\pi^2}{6}$.

Methods similar to the above generalize to find $\zeta(2k)$ for all k. However, it is unknown how to find the exact values of $\zeta(2k+1)$. If you could find a method for this, you would achieve great mathematical fame!

2 Gaussian Integral

The integral $\int e^{-x^2}\, dx$ has numerous applications to Physics, Statistics, Chemistry, and many other fields. While this integral has no elementary antiderivative, it can be evaluated in special cases, as we shall see below.

(a) Let $I := \int_{-\infty}^{\infty} e^{-x^2}\, dx$ – the Gaussian Integral. Explain why

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right)$$

(b) Rewrite the integral from the previous part as a single double integral.

(c) Use a change of variables to show that $I = \sqrt{\pi}$.

Evaluation

Complete the following survey by rating each problem. Each area will be rated on a scale of 1 to 5. For interest, 1 is “mind-numbing" while a 5 is “mind-blowing". For difficulty, 1 is “trivial/routine" while 5 is “brutal." For learning, 1 means “nothing new" while 5 means "profound awakening". Then you to estimate the amount of time you spent on each problem (in minutes).

	Interest	Difficulty	Learning	Time Spent
Basel Problem				
Gaussian Integral				

Indicate whether you believe this project was helpful in mastering the course material and/or if it was helpful in developing a deeper understanding of the subject. Also, indicate whether you think this project should be given to future Calculus III students.

	Yes	No
Helpful for the Course		
Helpful in Learning the Subject		
Assign Again		

Finally, you may write any comments, thoughts, or suggestions in the space below.