Kepler's Law

In 1609, based on data collected by Tycho Brahe, Johannes Kepler published a two laws about the motions of planets. The third would come later in 1919. Though Kepler himself never numbered these laws, they greatly strengthened Nicolaus Copernicus' heliocentric model of the Solar System. Later, Sir Isaac Newton would use these laws in the development of his laws of gravitation. We still use what are known as Kepler's Three Laws today. In this project, we will derive Kepler's Laws.

1 Kepler's First Law

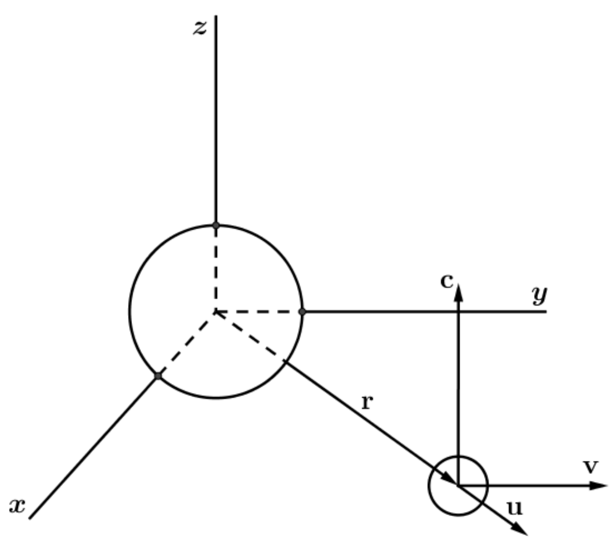
The First Law: The orbit of a planet is an ellipse with the Sun at one of the two foci.

(a) First, we need to show that the motion of a planet is planar and that the Sun lies in the plane of the planet's motion.

  (i) Let P be the point for the planet's center of mass and O be the point for the Sun's center of mass. Let $\mathbf{r}(t) = \vec{OP}$ and $\mathbf{v} = \mathbf{r}'(t)$. Show that $\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{r} \times \mathbf{a}$.

  (ii) Newton's Law of Gravitation states $\mathbf{F} = -\dfrac{GMm}{r^2}\mathbf{u}$, where G is the universal gravitation constant, M is the mass of the Sun, m is the mass of the planet, and $\mathbf{u}$ is a unit vector pointing from the Sun to the planet. Newton's Second Law states that $\mathbf{F} = m\,\mathbf{a}$. Use these two laws to find $\mathbf{a}$ in terms of $\mathbf{r}$.

  (iii) Use the previous parts to show that $\mathbf{r} \times \mathbf{v}$ is constant by showing its derivative is 0.



(b) Since $\mathbf{r} \times \mathbf{v}$ is constant (by the previous part), let $\mathbf{c} = \mathbf{r} \times \mathbf{v}$ be this constant vector. Let c denote the length of $\mathbf{c}$. Then we can write $\mathbf{c} = c\,\mathbf{k}$. Similarly, we can write $\mathbf{r} = r\,\mathbf{u}$, where r is the length of $\mathbf{r}$. Use the fact that $\mathbf{v} = \frac{d}{dt}(r\,\mathbf{u})$ and the definition of $\mathbf{c}$ to show that

$$\mathbf{c} = r^2\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right)$$

(c) Use the previous parts and the fact that $(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \cdot \mathbf{z})\,\mathbf{y} - (\mathbf{y} \cdot \mathbf{z})\,\mathbf{x}$ to show that

$$\mathbf{a} \times \mathbf{c} = \frac{d}{dt}(GM\,\mathbf{u})$$

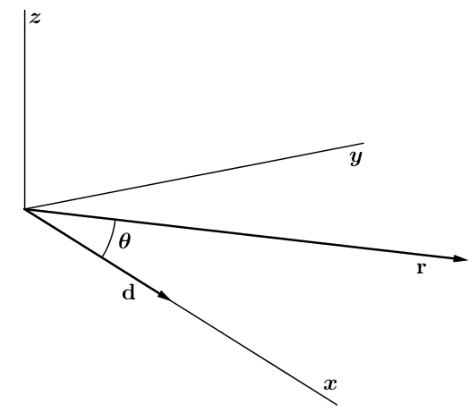
(d) Now use the fact that $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ to show that

$$\mathbf{a} \times \mathbf{c} = \frac{d}{dt}(\mathbf{v} \times \mathbf{c})$$

(e) Since the two different computations for $\mathbf{a} \times \mathbf{c}$ must agree, show that

$$\mathbf{v} \times \mathbf{c} = GM\mathbf{u} + \mathbf{d}$$

where $\mathbf{d}$ is a constant vector. Explain why $\mathbf{d}$ must lie in the xy–plane.



(f) Since $\mathbf{d}$ is constant, we can adjust coordinates via rotation so that $\mathbf{d}$ lies in the $\mathbf{i}$–direction, i.e. so that $\mathbf{d} = d\,\mathbf{i}$, where d is the length of $\mathbf{d}$. Let θ be the angle between $\mathbf{r}$ and $\mathbf{d}$. Show that $\mathbf{u} \cdot \mathbf{d} = d\cos\theta$.

(g) Use the previous parts and the fact that $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = (\mathbf{y} \times \mathbf{z}) \cdot \mathbf{x} = (\mathbf{z} \times \mathbf{x}) \cdot \mathbf{y}$ to show that

$$c^2 = r\,\mathbf{u} \cdot (GM\,\mathbf{u} + \mathbf{d})$$

and hence $c^2 = GMr + rd\cos\theta$.

(h) Show that

$$r = \frac{c^2}{GM + d\cos\theta} = \frac{c^2/GM}{1 + (d/GM)\cos\theta}$$

(i) For convenience, let $p = c^2/GM > 0$ and $e = d/GM$. Show that

$$r = p - ex$$

(j) Using the fact that $r^2 = x^2 + y^2$, show that

$$(1 - e^2)x^2 + 2pex + y^2 = p^2$$

so that we have an ellipse if $0 < |e| < 1$, a parabola if $e = \pm 1$, and a hyperbola if $|e| > 1$. There is no *analytic* way of determining which the orbit is. However since we know that planets repeat their orbit year after year, the orbit must then be elliptical.

(k) Show that this equation is equivalent to

$$\frac{(x + pe/(1 - e^2))^2}{p^2/(1 - e^2)^2} + \frac{y^2}{p^2/(1 - e^2)} = 1$$

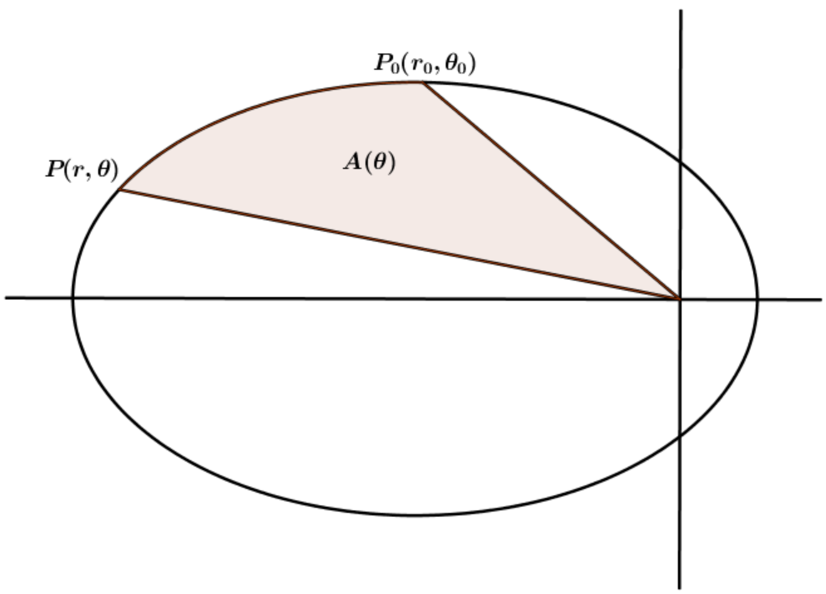
(l) For this ellipse, find the center and the length of the semimajor axis and semiminor axis. Using the fact that the square of the distance between the foci and the center is the difference of their squares, show that this distance is

$$\frac{p\,|e|}{1 - e^2}$$

and hence one foci is at the center, i.e. the location of our Sun.

2 Kepler's Second Law

The Second Law: During equal intervals of time, the line segment connecting a planet to the Sun sweeps out equal areas in equal times.



(a) Let $P(r, \theta)$ denote a point in a planets orbit and P_0 denote a fixed point in a planet's orbit. Furthermore, let $A(\theta) = \dfrac{1}{2}\displaystyle\int_{\theta_0}^{\theta} r^2\, d\theta$. Note that both r and θ depend on time, t. Use the Chain Rule and Fundamental Theorem of Calculus to show that

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

(b) We know that $\mathbf{u} = \frac{1}{r}\,\mathbf{r}$ and $\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$. Hence, show that

$$\mathbf{c} = r^2\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right) = r^2\frac{d\theta}{dt}\,\mathbf{k}$$

(c) Use the previous parts to show that

$$\frac{dA}{dt} = \frac{1}{2}\,c$$

Explain why this proves Kepler's Second Law.

3 Kepler's Third Law

The Third Law: The square of the orbital period for a planet is proportional to the cube of the semimajor axis for its orbit.

(a) Let T denote the orbital period, a the length of the semimajor axis, and b the length of the semiminor axis. The area of the ellipse formed by the orbit is πab. Use this to show that

$$\pi ab = \frac{cT}{2}$$

[Hint: Represent the area as an integral using the Fundamental Theorem of Calculus and use Kepler's Second Law.]

(b) We know that $b^2 = p^2/(1-e^2)$. Show that $b^2 = pa$ and $p = c^2/GM$.

(c) Show that $T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$.

(d) What is the constant of proportionality given by Kepler's Third Law? Is this constant the same for each planet?

Evaluation

Complete the following survey by rating each problem. Each area will be rated on a scale of 1 to 5. For interest, 1 is “mind-numbing" while a 5 is “mind-blowing". For difficulty, 1 is “trivial/routine" while 5 is “brutal." For learning, 1 means “nothing new" while 5 means "profound awakening". Then you to estimate the amount of time you spent on each problem (in minutes).

	Interest	Difficulty	Learning	Time Spent
Kepler’s First Law				
Kepler’s Second Law				
Kepler’s Third Law				

Indicate whether you believe this project was helpful in mastering the course material and/or if it was helpful in developing a deeper understanding of the subject. Also, indicate whether you think this project should be given to future Calculus III students.

	Yes	No
Helpful for the Course		
Helpful in Learning the Subject		
Assign Again		

Finally, you may write any comments, thoughts, or suggestions in the space below.