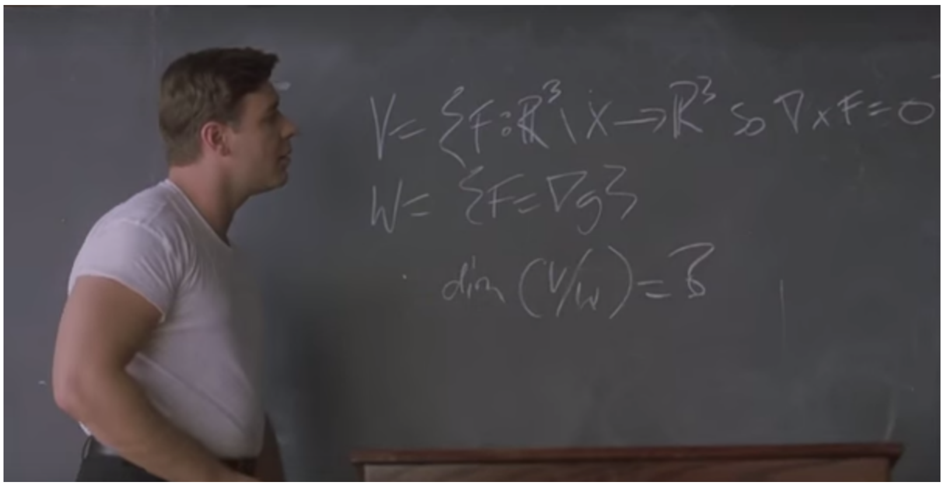
Beautiful Mind Problem

1 Introduction

In the Ron Howard film *A Beautiful Mind*, John Nash (played by Russell Crowe) gives his students a problem and says, "As I was saying, this problem here will take some of you many months to solve. For others among you, it will take you the term of your natural lives."



Nash's Problem:[1]

$$
\begin{aligned}
V &= \{F : \mathbb{R}^3 \setminus X \to \mathbb{R}^3 \colon \nabla \times F = \mathbf{0}\} \\
W &= \{F \colon F = \nabla g\} \\
\dim(V/W) &= 8
\end{aligned}
$$

Let's clear the meaning of the notation a bit. The set V is the set of vector fields defined except on a set X such that they are curl free. The set W is the set of vector fields that are the gradient of some function, i.e. conservative vector fields. The $\dim(V/W)$ will be explained in the next section but is the question portion: find a set X such that there are 'only eight' vector fields which are curl free but not a gradient field. Despite Nash's threat, let's solve this problem in a time better measured in minutes or hours rather than in months or years.

2 Linear Independence and Dimension

We say a collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent if and only if when $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, then $a_i = 0$ for all i. If a collection of vectors is not independent, we say that the collection of vectors is linearly dependent. For example, the collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \binom{1}{0}$ and $\mathbf{v}_2 = \binom{0}{1}$, is linearly independent as given

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

[1] There are some that say what is written is $\dim(V/W) = ?$. This is an equally valid question – though more difficult. However, solving the problem assuming that it is $\dim(V/W) = 8$ is not only easier, it shows how to solve the case where $\dim(V/W) = ?$.

Then $a_1 = a_2 = 0$. However, the collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix}$, is linearly dependent as taking $a_1 = 2$, $a_2 = -3$, and $a_3 = 1$, we have

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = 2\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + (-3)\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(a) Determine if the vectors $\mathbf{v}_1 = \binom{1}{3}$ and $\mathbf{v}_2 = \binom{3}{-1}$ are linearly independent or linearly dependent.

(b) Determine if the vectors $\mathbf{v}_1 = \binom{-2}{6}$ and $\mathbf{v}_2 = \binom{1}{-3}$ are linearly independent or linearly dependent.

(c) Determine if the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 2 \\ -4 \\ 9 \end{pmatrix}$ are linearly independent or linearly dependent.

(d) Determine if the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ are linearly independent or linearly dependent.

Now we say that a collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans (or generates) a space if every vector $\mathbf{v}$ can be written as

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

This is just a 'fancy' way of saying given any vector in our space (the space is 'probably' $\mathbb{R}^2$ or $\mathbb{R}^3$), we can find some linear combination of vectors from our collection that will give that vector. If the collection $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ not only spans the space but also is linearly independent, we say that the set is a basis for the space. The dimension of a space is the number of vectors in a basis. [It is not immediately obvious, but this number is unique and can be infinite.] For example, the space $\mathbb{R}^2$ has dimension 2 since given any vector $\binom{a_1}{a_2} \in \mathbb{R}^2$, we have

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 \end{pmatrix} = a_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then the collection $\{\binom{1}{0}, \binom{0}{1}\}$ is a basis for $\mathbb{R}^2$ (we just showed that it spans $\mathbb{R}^2$ and we showed before that the vectors are independent). Therefore, $\dim \mathbb{R}^2 = 2$. Similarly, $\dim \mathbb{R}^3 = 3$ as $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$. Indeed, the dimension of $\mathbb{R}^n$ is n.

These spaces do not have to be a set of vectors; they could be a set of functions. Consider P_3 over the reals – the set of polynomials of at most degree 3; that is, P_3 is the set of polynomials of degree

0, degree 1, degree 2, and degree 3. Then the set $\{\mathbf{1}, x, x^2, x^3\}$ is clearly linearly independent. Furthermore, every polynomial $a_0 + a_1x + a_2x^2 + a_3x^3$ can be written as a combination of these 'vectors' so that $\dim P_3 = 4$. In general, $\dim P_n = n + 1$.

Returning to Nash's question, recall V is the set of vector fields defined everywhere in $\mathbb{R}^3$ (except perhaps on X) which are curl free and W is the set of gradient fields. The notation V/W means 'modding out by W'. This is more technical and we will not rigorously define what this means. Essentially, it means we treat the vector fields in V which are gradient fields as '0' (meaning the zero vector field, i.e. $\mathbf{F}(x, y, z) = \mathbf{0}$ for all x, y, z). We now have enough language to understand the problem. Nash's problem is to find a set of points X so that the vector fields (defined everywhere except maybe at the points of X) which are curl free but are *not* gradient fields has dimension 8, i.e. $\dim(V/W) = 8$. The fact that $\dim(V/W) = 8$ means there are eight vector fields $\{\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_8\}$ so that each of the $\mathbf{F}$'s have the following properties:

(i) The $\mathbf{F}$'s are in V: $\nabla \times \mathbf{F}_i = \mathbf{0}$ for any point $(x, y, z) \in \mathbb{R}^3$ that is not in X.

(ii) The set of $\mathbf{F}_i$'s are linearly independent: No sum of multiplies of the $\mathbf{F}$'s is a gradient field, i.e. is conservative. That is, there are no numbers $a_1, a_2, \ldots, a_8$ and vector field F so that

$$\nabla \mathbf{F} = a_1\mathbf{F}_1 + a_2\mathbf{F}_2 + \cdots + a_8\mathbf{F}_8$$

(iii) The set of $\mathbf{F}_i$'s generate V/W: If $\mathbf{G}$ is a vector field with $\nabla \times \mathbf{G} = \mathbf{0}$, then there are numbers $a_1, a_2, \ldots, a_8$ and a function field $\mathbf{F}$ so that

$$\nabla \mathbf{F} = \mathbf{G} - (a_1\mathbf{F}_1 + a_2\mathbf{F}_2 + \cdots + a_8\mathbf{F}_8)$$

3 2D Case

We first solve Nash's problem in the 2-dimensional case, where it is simpler. This will show us how the 3-dimensional case behaves. Let $X = \{(0, 0)\}$ be the origin, V be the set of vector fields $\mathbf{F}$ on $\mathbb{R}^2 \setminus X$ with $\nabla \times \mathbf{F} = \mathbf{0}$, and W the set of vector fields $\mathbf{F}$ which are gradient fields. We will show that $\dim(V/W) = 1$. So we must find a single vector field $\mathbf{F}$ satisfying (i)–(iii) from above. We will show that

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

satisfies these properties.

(a) Show that $\nabla \times \mathbf{F} = \mathbf{0}$ so that $\mathbf{F}$ satisfies (i).

(b) Show that $\oint_C a\mathbf{F} \cdot d\mathbf{r} \neq 0$ for any $a \neq 0$, where C is the unit circle centered at the origin, oriented counterclockwise. Explain why this implies that $a\mathbf{F}$ is not conservative, i.e. that $\mathbf{F}$ is not a gradient field showing that $\mathbf{F}$ satisfies (ii).

(c) Suppose $\mathbf{G}$ is a vector field defined everywhere except for possible the origin with $\nabla \times \mathbf{G} = \mathbf{0}$. Suppose C and C' are two different closed paths not passing through the origin so that C' can be deformed to C without passing through the origin. [Imagine the paths are strings. You want to be able to stretch, shrink, bend, etc C' to become C without cutting or tearing C' or passing through the origin.] Use Green's Theorem to argue (or show directly) that

$$\oint_C \mathbf{G} \cdot d\mathbf{r} = \oint_{C'} \mathbf{G} \cdot d\mathbf{r}$$

(d) If C is a simple closed curve which does not pass through the origin, explain how one can deform C (as in the previous part) so that it is a circle that wraps (clockwise or counterclockwise) around the origin k times for some number k.

(e) Let C denote the unit circle and assume $\nabla \times \mathbf{G} = \mathbf{0}$. If C' is a curve which wraps around the origin (without passing through it) k times. Let $I = \oint_C \mathbf{G} \cdot d\mathbf{r}$. Show that

$$\oint_{C'} \mathbf{G} \cdot d\mathbf{r} = kI$$

$$\oint_{C'} \mathbf{F} \cdot d\mathbf{r} = k$$

(f) Now define for any vector field $\mathbf{G}$, a vector field $\mathbf{H}$ by $\mathbf{H} := \mathbf{G} - \frac{I}{2\pi}\mathbf{F}$. Show that $\mathbf{H}$ has the closed loop property:

$$\oint_C \mathbf{H} \cdot d\mathbf{r} = 0$$

where C is any simple closed curve. Explain why this shows that $\mathbf{H}$ is conservative. Explain why this shows that (iii) holds for $\mathbf{F}$.

(g) Observe that when we remove a single point – namely $(0,0)$ – we have $\dim(V/W) = 1$. Make a prediction of what $\dim(V/W)$ would be if we removed, say, 8 distinct points. Explain this result 'geometrically'; that is, what 'geometric' measurement is $\dim(V/W)$ making?

4 3D Case

Now we have solved the 2-dimensional case where X is any collection of distinct points. One can quickly generalize (with some work) to find $\dim(V/W)$ when X is arbitrary. The solution for the 2-dimensional cases generalizes to the 3-dimensional problem at hand.

(a) Suppose that $X = \{(0,0,0)\}$. Use Stokes' Theorem to argue that $\dim(V/W) = 0$; that is, show that every irrotational (curl free) vector field defined on $\mathbb{R}^3 \setminus X$ is a conservative vector field. In this case, what is $\dim(V/W)$? How does this differ than before? Can $\dim(V/W)$ be measuring the same 'geometric' properties?

(b) Suppose that X is the z–axis in $\mathbb{R}^3$. Carefully use the work from the 2-dimensional case to show that $\dim(V/W) = 1$.

(c) Interpret the previous part 'geometrically'; that is, what 'geometric' measurement is $\dim(V/W)$ making?

(d) Using the ideas of the previous part, find X so that $\dim(V/W) = 8$. Consider the 'harder' version of the problem: given a set X, what is $\dim(V/W)$?

Note that all of these calculations and intricate relationships between the 'geometry' of the space and functions on the space are simple examples of de Rham cohomology. These are objects of interest in the fields Algebraic Topology and Differential Geometry – among others. However, Algebraic Topology and Differential Geometry are only a few of the many fields of Mathematics which study the relationship between function on a space and the 'geometry' of the space.

Bibliography

A Beautiful Mind. Ron Howard. Universal Pictures, 2002. Film.

Evaluation

Complete the following survey by rating each problem. Each area will be rated on a scale of 1 to 5. For interest, 1 is “mind-numbing" while a 5 is “mind-blowing". For difficulty, 1 is “trivial/routine" while 5 is “brutal." For learning, 1 means “nothing new" while 5 means "profound awakening". Then you to estimate the amount of time you spent on each problem (in minutes).

	Interest	Difficulty	Learning	Time Spent
Linear Independence and Dimension				
2D Case				
3D Case				

Indicate whether you believe this project was helpful in mastering the course material and/or if it was helpful in developing a deeper understanding of the subject. Also, indicate whether you think this project should be given to future Calculus III students.

	Yes	No
Helpful for the Course		
Helpful in Learning the Subject		
Assign Again		

Finally, you may write any comments, thoughts, or suggestions in the space below.