The Sunshine Formula

Opening most weather applications, you will see the sunrise and sunset times for the next few days. How are these calculated? What may seem like a simple prediction actually requires quite a lot of advanced Mathematics and Physics. In order to make truly accurate long term predictions, one would have to take into account relativity. Even in this case, one must continuously update the model to keep the predicted times accurate. In this project, we will use a simple vector approach to predict daily sunrise times that can be a good approximation over short time periods.

(a) Consider two nonparallel vectors $\mathbf{l}$ and $\mathbf{r}$ with the same point of origin. Sketch the result of rotating $\mathbf{r}$ about $\mathbf{l}$. What surface does this create?

(b) Assume that the rotation occurs at a uniform rate, making a complete revolution in T units of time. Then $\mathbf{r}$ is a function of time. Write $\mathbf{r} = \sigma(t)$ for some function $\sigma(t)$. Define the initial position of $\mathbf{r}(t)$ to be $\mathbf{r}_0 = \sigma(0)$. Let λ be the angle between $\mathbf{r}$ and $\mathbf{l}$. Since $\mathbf{r}$ and $\mathbf{l}$ are not parallel, we know $0 < \lambda < \pi$. Let $\mathbf{m}_0$ be a unit vector, perpendicular to $\mathbf{l}$, and lying in the same plane as $\mathbf{r}_0$ and $\mathbf{l}$. Show that

$$\mathbf{r}_0 = \cos\lambda\,\mathbf{l} + \sin\lambda\,\mathbf{m}_0$$

(c) Show that

$$\mathbf{m}_0 = \frac{1}{\sin\lambda}\,\mathbf{r}_0 - \frac{\cos\lambda}{\sin\lambda}\,\mathbf{l}$$

and use this definition to confirm that $\mathbf{m}$ and $\mathbf{l}$ are perpendicular.

(d) Now define $\mathbf{n}_0 = \mathbf{l} \times \mathbf{m}_0$. Then $\mathbf{l}$, $\mathbf{m}_0$, and $\mathbf{n}_0$ are all mutually orthogonal unit vectors. We can equally consider these vectors as rotating about $\mathbf{l}$ so that we can consider these vectors as functions of time: $\mathbf{n}(t)$ and $\mathbf{m}(t)$. Explain why we have

$$\mathbf{m}(t) = \frac{1}{\sin\lambda}\,\mathbf{r} - \frac{\cos\lambda}{\sin\lambda}\,\mathbf{l}$$

(e) Since $\mathbf{m}$ is perpendicular to $\mathbf{l}$, it rotates in a circle in the plane formed by $\mathbf{m}_0$ and $\mathbf{n}_0$. It rotates through an angle of 2π in time T. Then it goes through an angle of $2\pi t/T$ in t units of time. Then $\mathbf{m} = \cos\left(\frac{2\pi t}{T}\right)\mathbf{m}_0 + \sin\left(\frac{2\pi t}{T}\right)\mathbf{n}_0$. Show that

$$\mathbf{r}(t) = \cos\lambda\,\mathbf{l} + \sin\lambda\cos\left(\frac{2\pi t}{T}\right)\mathbf{m}_0 + \sin\lambda\sin\left(\frac{2\pi t}{T}\right)\mathbf{n}_0$$

(f) The previous part gives us a formula for $\mathbf{r} = \sigma(t)$ in terms of t so that we know how $\mathbf{r}$ changes in time.

(i) Find $\mathbf{l}\cdot\mathbf{r}$.

(ii) Find $|\mathbf{l}\times\mathbf{r}_0|$.

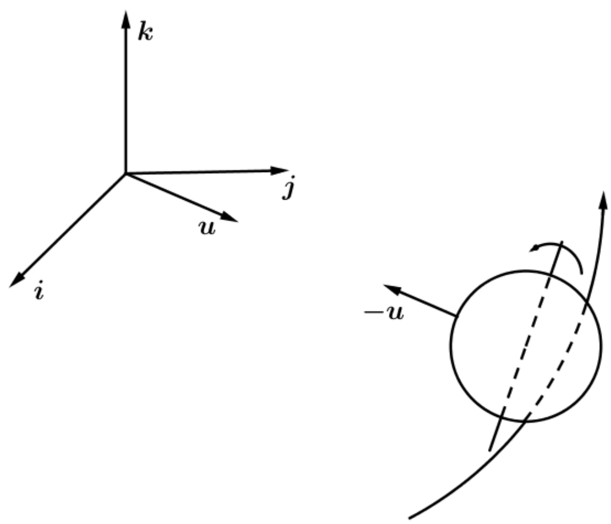
(iii) Explain why $\mathbf{n}_0 = \frac{\mathbf{l}\times\mathbf{r}_0}{|\mathbf{l}\times\mathbf{r}_0|}$ and find $\mathbf{l}\times\mathbf{r}_0$ in terms of $\mathbf{n}_0$ and λ. Show $\sin\lambda\,\mathbf{m}_0 = \mathbf{r}_0 - (\mathbf{r}_0\cdot\mathbf{l})\mathbf{l}$.

(iv) Show that

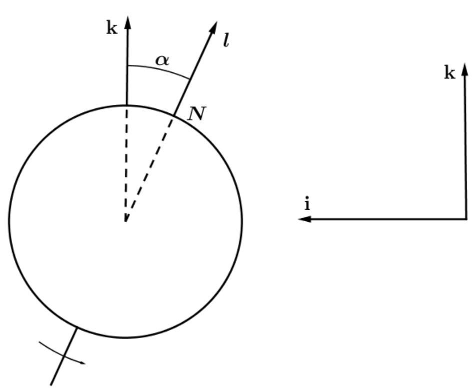
$$\mathbf{r} = (\mathbf{r}_0 \cdot \mathbf{l})\,\mathbf{l} + \cos\left(\frac{2\pi t}{T}\right)(\mathbf{r}_0 - (\mathbf{r}_0 \cdot \mathbf{l})\,\mathbf{l}) + \sin\left(\frac{2\pi t}{T}\right)(\mathbf{l} \times \mathbf{r}_0)$$

(g) Show that the angular velocity of the tip of $\mathbf{r}$ is $\dfrac{2\pi}{T}\sin\lambda$. [Recall $v = \frac{\text{distance}}{\text{time}}$.]

(h) Now we shift to a Sun–Earth system. We assume that the Sun is a fixed origin of our coordinate system with the Earth moves about the Sun with uniform speed and in a circular motion.[1] Let $\mathbf{u}$ be a unit vector pointing from the Sun to the Earth and T_y be the length of the year. Find an equation for the vector $\mathbf{u}$ at time t.

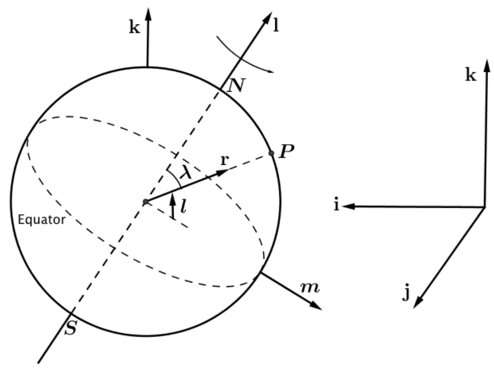


(i) Now we have to factor in the rotation of the Earth. Now we assume the Earth rotates about an axis $\mathbf{l}$. We assume that $\mathbf{l}$ is fixed with respect to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The tilt of the Earth (the angle $\mathbf{l}$ makes with $\mathbf{k}$) is (currently) approximately $\alpha = 23.5°$.



[1]Most people are aware that the Earth's orbit is not circular but is rather slightly elliptical. However, this will not make a difference for our model. For one, the Earth's orbit is only slightly elliptical. But more importantly, this foci of this ellipse move over time. This results from Einstein's General Theory of Relativity. In addition, the Earth itself is not a sphere and the tilt of the Earth precesses over time. All these factors greatly affect the angle of the sun. However, to compensate for all these factors makes the model incredibly complicated and need numerical techniques. We use a simplified model to get an idea of a real method of deriving the Sunrise 'Formula'.

We choose to measure time so that $t = 0$ is the first day of summer in the northern hemisphere (June 21st). Then $\mathbf{l} = -\mathbf{i}$ when $t = 0$ so that $\mathbf{l} = \cos(\alpha)\,\mathbf{k} - \sin(\alpha)\,\mathbf{i}$. Let $\mathbf{r}$ be the unit vector at time t from the center of the Earth to a fixed point P on the Earth's surface. We choose P so that it is noon at P when $t = 0$.



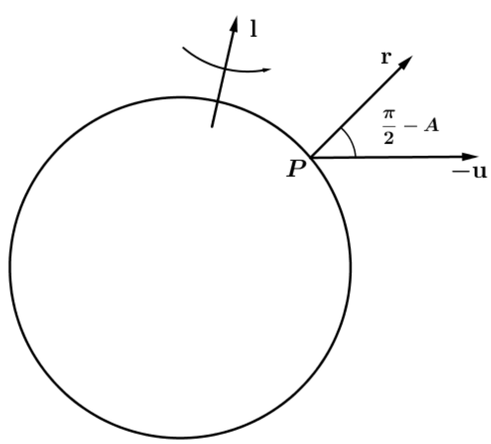
Then $\mathbf{r}$ lies in the plane spanned by $\mathbf{i}$ and $\mathbf{l}$. Now let $\mathbf{m}_0 = -(\sin\alpha)\,\mathbf{k} - \cos(\alpha)\,\mathbf{i}$ be the unit vector orthogonal to $\mathbf{l}$. Then $\mathbf{r}_0 = \cos\lambda\,\mathbf{l} + \sin\lambda\,\mathbf{m}_0$, where λ is the angle between $\mathbf{l}$ and $\mathbf{r}_0$. But $\lambda = \frac{\pi}{2} -$, where l is the latitude of the point P. Then $\mathbf{r}_0 = \sin l\,\mathbf{l} + \cos l\,\mathbf{m}_0$. Let $\mathbf{n}_0 = \mathbf{l} \times \mathbf{m}_0$ and show that $\mathbf{n}_0 = -\mathbf{j}$.

(j) Let T_d denote the time of day at the point P. Use the previous parts to show that

$$\mathbf{r} = -\left[\sin l \sin\alpha + \cos l \cos\alpha \cos\left(\frac{2\pi t}{T_d}\right)\right]\mathbf{i} - \cos l \sin\left(\frac{2\pi t}{T_d}\right)\mathbf{j} + \left[\sin l \cos\alpha - \cos l \sin\alpha \cos\left(\frac{2\pi t}{T_d}\right)\right]\mathbf{k}$$

(k) Use the previous parts to find the speed of rotation due to the Earth's rotation in Syracuse, NY. [Note that $T_d = 23.934$ hours $R = 6371$ km and the latitude of Syracuse is $43.05°$.]

(l) Now let A denote the angle of elevation of the sun above the horizon. From the diagram below, we can see that $\sin A = -\mathbf{u} \cdot \mathbf{r}$.



Then expanding the product, we have

$$\sin A = \cos\left(\frac{2\pi t}{T_y}\right)\sin l \sin\alpha + \cos l\left[\cos\left(\frac{2\pi t}{T_y}\right)\cos\alpha\cos\left(\frac{2\pi t}{T_d}\right) + \sin\left(\frac{2\pi t}{T_y}\right)\sin\left(\frac{2\pi t}{T_d}\right)\right]$$

We want to find how $\sin A$ varies throughout a day (this will give how A varies throughout the day). Explain why $\cos(2\pi t/T_y)$ and $\sin(2\pi t/T_y)$ are approximately constant.

(m) Let n be the nth day from June 21st. Then we have

$$\sin A = P\sin l + \cos l\left[Q\cos\left(\frac{2\pi t}{T_d}\right) + R\sin\left(\frac{2\pi t}{T_d}\right)\right]$$

where $P = \cos(2\pi n/364)\sin\alpha$, $Q = \cos(2\pi n/365)\cos\alpha$, $R = \sin(2\pi n/365)$. Find U such that $U\cos(2\pi(t - t_n)/T_d)$ is the same as the expression in brackets above. [Hint: You may want to make use of $\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$ to expand $U\cos(2\pi(t - t_n)/T_d)$.]

(n) Let τ denote the time in hours from noon on the nth day so that $(t - t_n)/T_d = \tau/24$. Then we have

$$\sin A = \sin l\cos\left(\frac{2\pi n}{365}\right)\sin\alpha + \cos l\cos\left(\frac{2\pi\tau}{24}\right)\sqrt{1 - \cos^2\left(\frac{2\pi n}{365}\right)\sin^2\alpha}$$

Find the angle of elevation of the Sun in Syracuse, NY July 12, 2017 at 3:23pm. Use a computer system to find the sunrise time for Syracuse, NY on August 1, 2017.

Evaluation

Complete the following survey by rating each problem. Each area will be rated on a scale of 1 to 5. For interest, 1 is “mind-numbing" while a 5 is “mind-blowing". For difficulty, 1 is “trivial/routine" while 5 is “brutal." For learning, 1 means “nothing new" while 5 means "profound awakening". Then you to estimate the amount of time you spent on each problem (in minutes).

	Interest	Difficulty	Learning	Time Spent
Parts (a)–(d)				
Parts (e)–(i)				
Parts (j)–(n)				

Indicate whether you believe this project was helpful in mastering the course material and/or if it was helpful in developing a deeper understanding of the subject. Also, indicate whether you think this project should be given to future Calculus III students.

	Yes	No
Helpful for the Course		
Helpful in Learning the Subject		
Assign Again		

Finally, you may write any comments, thoughts, or suggestions in the space below.